

A Presentation of the Absolute Galois Group of $\mathbb{Q}_2$

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Abstract

We give an explicit profinite presentation of the absolute Galois group $G_{\mathbb{Q}_2}$. The presentation has four generators σ, τ, x_0, x_1 ; the closed normal subgroup generated by x_0, x_1 is required to be pro-2; and there are two defining relations. The first is the tame relation $\tau^\sigma = \tau^2$. The second, involving the pro-2 generators, is the explicit relation stated in Theorem 1.2. This is a dyadic counterpart to the known four-generator presentations of $G_{\mathbb{Q}_p}$ for odd primes p . The case $p = 2$ requires a separate argument because the odd-prime formula contains the exponent $(p + 1)/2$, which is not integral at $p = 2$.

The proof first identifies the tame quotient and the maximal pro-2 quotient of the proposed group, together with their common map to the unramified quotient $\mathbb{Z}_2$. The maximal pro-2 quotient is identified with the rank-three Demushkin group attached to $\mathbb{Q}_2$ using Labute's classification, local reciprocity, and a lifting theorem for automorphisms of its abelianization. We then compare lifting problems from the proposed group and from $G_{\mathbb{Q}_2}$ through extensions with finite elementary abelian 2-group kernels. On the presentation side these lifting problems are controlled by an explicit three-term complex obtained from the two defining relators; on the Galois side they are controlled by continuous cohomology and local Tate duality. For self-dual simple $\mathbb{F}_2$ -modules, the lifting count contains an additional quadratic obstruction; we compute it from the dyadic filtration of square classes and an explicit formula for the cohomological norm in a quadratic extension. Finite Fourier inversion and induction on the order of the normal 2-subgroup in a finite target then give equality of epimorphism counts to every finite group. A profinite reconstruction argument yields the asserted presentation.

1 Introduction and main theorem

This section fixes the conventions that determine the two displayed relators, states the presentation theorem, and isolates the marked local normalization needed in its proof. The final subsection reduces the remaining argument to a comparison of finite epimorphism counts.

For odd primes p , the absolute Galois group of $\mathbb{Q}_p$ admits an explicit four-generator presentation of Jannsen–Wingberg type; see [1, Chapter VII]. The dyadic case is exceptional because the odd-prime formula uses an exponent containing $(p + 1)/2$. We give a separate dyadic relation.

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Throughout, exponentiation by an element of $\widehat{\mathbb{Z}}$ has its usual profinite meaning. Let

$$\omega_2 = (1 \text{ in } \mathbb{Z}_2, 0 \text{ in } \mathbb{Z}_\ell \text{ for every odd prime } \ell).$$

Thus $\omega_2^2 = \omega_2$.

We use the conventions

$$x^g = g^{-1}xg, \quad [x, y] = x^{-1}y^{-1}xy.$$

The unramified coordinate is normalized geometrically throughout: if Φ_{ar} is arithmetic Frobenius and $\Phi_{\text{geom}} = \Phi_{\text{ar}}^{-1}$, then $\nu_{\text{ur}}(\Phi_{\text{geom}}) = 1$. With the right-conjugation convention above, a standard tame generator satisfies

$$\tau^{\Phi_{\text{geom}}} = \tau^q.$$

Thus the displayed relation $\tau^\sigma = \tau^2$ means that σ maps to geometric Frobenius. Put

$$\sigma_2 = \sigma^{\omega_2}, \quad u_i = (x_i \tau)^{\omega_2} \ (i = 0, 1), \quad d_0 = u_0 x_0^{-1}, \quad (1.1)$$

$$z_0 = x_0^{\sigma_2}, \quad c_0 = [d_0, z_0], \quad g_0 = \sigma_2^2, \quad d_g = d_0^{g_0}, \quad (1.2)$$

$$h_c = [d_g, d_0], \quad h_0 = x_0^{g_0} x_0 d_g d_0 d_0^2 h_c. \quad (1.3)$$

Proposition 1.1 (Marked dyadic Demushkin normalization). *Let $D_{\text{loc}} = G_{\mathbb{Q}_2}(2)$ and let $\nu_{\text{ur}} : D_{\text{loc}} \rightarrow \mathbb{Z}_2$ be normalized by sending geometric Frobenius to 1. There exist topological generators a, s, y such that*

$$D_{\text{loc}} \cong \langle a, s, y \mid a^2 s^4 [s, y] = 1 \rangle_{\text{pro-2}}, \quad \nu_{\text{ur}}(a) = -2, \quad \nu_{\text{ur}}(s) = 1, \quad \nu_{\text{ur}}(y) = 0. \quad (1.4)$$

The proof is given in Sections 3.1–3.3.

With the marked Demushkin normalization available, the main theorem takes the following form.

Theorem 1.2 (Presentation theorem). *The absolute Galois group $G_{\mathbb{Q}_2}$ is isomorphic to the profinite group topologically generated by*

$$\sigma, \tau, x_0, x_1,$$

for which the closed normal subgroup generated by x_0, x_1 is pro-2, and which is subject to the two relations

$$\tau^\sigma = \tau^2 \quad (1.5)$$

and

$$h_0 u_1^{-1} x_1^\sigma c_0 = 1, \quad (1.6)$$

where the auxiliary words are defined in (1.1)–(1.3).

Explicit presentations of this kind have a substantial history. For odd p , the structure of the full group $G_{\mathbb{Q}_p}$ by generators and relations is due to Jannsen and Wingberg [12]; see [1, Chapter VII] for a systematic account. In residue characteristic two, Zel'venskii [13] described the Galois group of the maximal 2-extension for 2-adic base fields of odd degree, and Diekert [14] obtained structure results for absolute Galois groups of dyadic fields under additional hypotheses. Presentations of local Galois groups also feed anabelian questions about their automorphisms; see for instance [16].

During preparation of this manuscript, we learned of independently proposed candidate presentations for $G_{\mathbb{Q}_2}$ by other researchers. Some may be isomorphic, after changes of generators, to the presentation in Theorem 1.2. At the time of writing, these candidate presentations and the proposed isomorphisms have not been fully validated.

Remark 1.3 (Role of the computational verification). A separate computational verification of the displayed word exists but is not reproduced here. It is independent of the proof below and is not used as an input to the marked Demushkin normalization or to any mathematical inference.

Write Γ_A for the profinite group in [Theorem 1.2](#). The condition that the normal closure of x_0, x_1 be pro-2 is part of its presentation. For an epimorphism to a finite group G , this condition is equivalent to requiring the images of x_0, x_1 to lie in the 2-core $O_2(G)$.

1.1 Strategy of the proof

The proof has six stages.

1. The candidate Γ_A is constructed from its admissible finite quotients. A reconstruction lemma reduces the theorem to equality of epimorphism counts from Γ_A and $G_{\mathbb{Q}_2}$ to every finite group.
2. The tame quotient of Γ_A is identified with the tame quotient of $G_{\mathbb{Q}_2}$. Its maximal pro-2 quotient is identified with $G_{\mathbb{Q}_2}(2)$ in a form preserving the full unramified character to $\mathbb{Z}_2$, not merely its reduction modulo 2.
3. The tame and maximal pro-2 maps are combined over their common unramified quotient. Fixing this fibre-product datum turns the comparison into a finite lifting problem with a prescribed tame quotient and a prescribed map to an elementary abelian 2-group.
4. For extensions with elementary abelian 2-group kernel, lifts from the candidate are governed by the two defining relators, while lifts from $G_{\mathbb{Q}_2}$ are governed by continuous cohomology. The two theories are shown to have the same obstruction dimensions, lift multiplicities, duality pairings, and connecting maps.
5. When a simple coefficient module is self-dual, a scalar central pushout produces a quadratic obstruction. An exact word calculation on the candidate side and the dyadic square-class filtration on the local side give the same quadratic Gauss sum.
6. For a finite target, the normal 2-subgroup is reduced through an elementary quotient and a central Frattini layer. Fourier inversion and the partition of homomorphisms according to their image subgroup reduce the count to strictly smaller targets. Strong induction gives equality of all finite epimorphism counts, and profinite reconstruction gives the required isomorphism.

The dependence of the argument is therefore

$$\begin{array}{c}
\text{marked tame and maximal pro-2 quotients} \\
\Downarrow \\
\text{common tame-pro-2 data over } \mathbb{Z}_2 \\
\Downarrow \\
\text{linear lifting comparison and quadratic obstruction comparison} \\
\Downarrow \\
\text{induction on the normal 2-subgroup of a finite target} \\
\Downarrow \\
|\mathrm{Sur}(\Gamma_A, G)| = |\mathrm{Sur}(G_{\mathbb{Q}_2}, G)| \quad \text{for every finite group } G \\
\Downarrow \\
\Gamma_A \cong G_{\mathbb{Q}_2}.
\end{array}$$

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1.2 What is new

We record the aspects of the argument that we believe are new.

The route to the presentation is enumerative: we match $|\mathrm{Sur}(\Gamma_A, G)| = |\mathrm{Sur}(G_{\mathbb{Q}_2}, G)|$ for every finite group G and conclude by a one-sided profinite reconstruction. To our knowledge, this counting comparison has not previously been used to establish a presentation of a local absolute Galois group; the known proofs for odd p [12] and the dyadic partial results [13], [14] proceed through embedding-problem solvability and the Demushkin classification.

The reconstruction lemma (Lemma 2.5) is stated in a one-sided form: P is assumed topologically finitely generated while Q is an arbitrary profinite group, with equality of the surjection counts $|\mathrm{Sur}(P, H)| = |\mathrm{Sur}(Q, H)|$ the only hypothesis required for $P \cong Q$. Classical finite-quotient rigidity [17] (see also [4, Section 2.5]) assumes finite generation on both sides, while for small profinite groups the set of finite quotients is already determining [18, Proposition 16.10.7]. Although both sides are known to be finitely generated in the present application, the one-sided form is the natural strength of the argument — it matches the counting data the proof actually produces — and it is the shape required in settings where finite generation of the Galois side is open, such as the restricted-ramification groups $G_{\mathbb{Q}, S}$, which are small by Hermite–Minkowski yet not known to be topologically finitely generated.

Lifting problems are controlled by an evaluated Fox–Jacobian three-term complex, with cup products computed in an explicit Heisenberg model. Free differential calculus is an established tool in pro- p Galois theory [20]; the device of evaluating the Jacobian on each finite target to control lifting counts appears to be new here. Finite Fourier inversion over character groups of finite modules is used to invert the lifting-count recursion — a classical enumerative device in the Frobenius tradition, applied inside a Galois-presentation argument.

For self-dual simple $\mathbb{F}_2$ -modules, the lifting count acquires a quadratic obstruction, computed through Arf invariants and the dyadic filtration of square classes: the count is not determined

by the linear duality data alone, in contrast with the odd- p pattern of [1, Chapter VII]. The dyadic obstruction is computed with tools imported from quadratic-form theory and topology: Arf invariants, Stiefel–Whitney classes [10], [6], the Evens norm from finite-group cohomology [9], [11], and Gauss-sign computations.

The boundary-framed marked target $\mathcal{Y} = (Y, L_Y, \pi_Y, \theta_Y)$ of Definition 4.1 and the exact-image count e_Γ^β are defined for an arbitrary profinite source with a tame-pro-2 boundary; nothing in the construction is specific to $\mathbb{Q}_2$. The framework is a candidate approach to presentations of G_K for other dyadic fields, where only partial results are known [14], [19].

The normalized-cochain ledger of Appendix 12 carries Shapiro maps, corestriction, and the Evens norm at the level of explicit normalized cochains with transversal bookkeeping, making transfer-level cohomology computations finite and machine-checkable; standard treatments [5], [9] work at the level of classes.

2 The candidate as a profinite group

Starting from the four generators, the two relators, and the requirement on the normal closure of x_0, x_1 , we construct the candidate as an inverse limit of finite quotients. The section ends with an exact finite-quotient semantics for this construction and the reconstruction criterion used in Section 10.

2.1 The marked quotient construction

Let F be the free profinite group on σ, τ, x_0, x_1 . The finite quotients relevant to the presentation are precisely those in which the four distinguished elements generate, satisfy both relators, and have 2-group wild normal closure. Formally, call an epimorphism

$$\varphi : F \twoheadrightarrow G$$

to a finite group *admissible* if the distinguished tuple generates G , satisfies (1.5) and (1.6), and the normal closure of $\varphi(x_0), \varphi(x_1)$ is a 2-group. Let $\mathcal{Q}_A$ be the class of all admissible finite quotients and set

$$N_A = \bigcap_{\varphi \in \mathcal{Q}_A} \ker \varphi, \quad \Gamma_A = F/N_A. \quad (2.1)$$

Let W_A denote the closed normal subgroup of Γ_A generated by x_0, x_1 .

Lemma 2.1 (Finite subdirect closure). *If $F \twoheadrightarrow G_i$ are finitely many admissible quotients and Y is the image of F in $\prod_i G_i$, then $F \twoheadrightarrow Y$ is admissible and its kernel is the intersection of the given kernels.*

Proof. The two relations hold coordinatewise. The distinguished tuple generates Y . The normal closure of x_0, x_1 in Y lies in

$$Y \cap \prod_i O_2(G_i),$$

which is a finite 2-group. Since Y is the image of the diagonal map $F \rightarrow \prod_i G_i$, one has

$$\ker(F \rightarrow Y) = \bigcap_i \ker(F \rightarrow G_i),$$

which proves the assertion about kernels. □

Two features make the construction usable in counting arguments: every finite quotient of the candidate is itself admissible, and epimorphisms to a finite group admit a purely equational description. We record them in turn.

Lemma 2.2 (Cofinality). *Every finite quotient of Γ_A is an admissible marked quotient. In particular, W_A is pro-2.*

Proof. Let $K \triangleleft_{\text{open}} F$ contain N_A . For every $g \notin K$, the definition of N_A gives an admissible quotient whose kernel does not contain g . Compactness of $F \setminus K$ gives finitely many such kernels whose intersection is contained in K . By Lemma 2.1, this intersection is itself the kernel of an admissible quotient. Therefore F/K is a quotient of an admissible quotient, and admissibility passes to quotients. The last assertion follows because every finite image of W_A is a 2-group. $\square$

Cofinality identifies which finite groups occur; the counting arguments also need the epimorphisms themselves in equational form.

Proposition 2.3 (Epimorphic finite-quotient semantics). *For every finite group G , epimorphisms $\Gamma_A \twoheadrightarrow G$ are in natural bijection with generating quadruples $(\sigma, \tau, x_0, x_1) \in G^4$ satisfying (1.5), (1.6), and $x_0, x_1 \in O_2(G)$.*

Proof. An epimorphism sends W_A to a normal 2-subgroup of G , hence into $O_2(G)$. Conversely, a generating quadruple with $x_0, x_1 \in O_2(G)$ has normal 2-group closure and defines an admissible quotient in $\mathcal{Q}_A$. $\square$

Remark 2.4. The corresponding statement for arbitrary, non-surjective homomorphisms into a fixed ambient group is false: the image of W_A is normal in the image subgroup, but need not lie in the 2-core of the ambient group. This is why the proof below keeps track of exact images and their induced markings.

2.2 A reconstruction lemma

Lemma 2.5 (One-sided profinite reconstruction). *Let P be a topologically finitely generated profinite group and let Q be any profinite group. If*

$$|\text{Sur}(P, H)| = |\text{Sur}(Q, H)|$$

for every finite group H , then $P \cong Q$.

Proof. We construct an epimorphism in each direction from compatible finite quotients and then use Hopficity to identify both maps as isomorphisms.

Since P is finitely generated, $\text{Sur}(P, H)$ and hence $\text{Sur}(Q, H)$ are finite for every finite H . The two groups have exactly the same finite quotients.

For $n \geq 1$, let $R_n(P)$ be the intersection of the kernels of all maps from P to groups of order at most n , and put $P_n = P/R_n(P)$. The P_n form a cofinal inverse system of finite quotients of P . Every P_n is a quotient of Q . For any finite set of indices, a sufficiently large P_m maps onto all corresponding P_n ; composing an epimorphism $Q \twoheadrightarrow P_m$ gives a compatible family on that finite set. Compactness of the inverse limit of the finite nonempty sets $\text{Sur}(Q, P_n)$ yields a compatible family and therefore an epimorphism $Q \twoheadrightarrow P$.

Interchanging the two groups, use the directed system of all finite quotients of Q . The sets of epimorphisms from P to those quotients are finite and have the finite intersection property, so there

is an epimorphism $P \twoheadrightarrow Q$. The composite $P \twoheadrightarrow Q \twoheadrightarrow P$ is a surjective endomorphism of a finitely generated profinite group. Such groups are Hopfian; see, for example, [4, Section 2.5]. Hence the composite is an isomorphism, and so is $P \twoheadrightarrow Q$. $\square$

The topological finite generation of $G_{\mathbb{Q}_2}$ required in this argument is classical; see [1, Theorem (7.5.14)] or [12]. Reconstruction of profinite groups from their finite quotients goes back to [17]; the counting refinement used here is a one-sided variant.

3 The tame and maximal pro-2 quotients

The candidate construction of Section 2 is now compared with the corresponding tame and maximal pro-2 quotients of $G_{\mathbb{Q}_2}$. Besides the abstract quotient identifications, we determine compatible maps to the full unramified quotient $\mathbb{Z}_2$; those marked maps are the input to the fibre product in Section 4.

Let

$$T_{\text{tame}} = \langle \sigma, \tau \mid \tau^\sigma = \tau^2 \rangle_{\text{prof}}$$

denote the finite-quotient tame group.

Lemma 3.1 (Finite tame quotients). *In every finite quotient generated by s, t with $t^s = t^2$, the element t has odd order, and the quotient has the form*

$$C_e \rtimes C_n, \quad e \text{ odd}, \quad s^{-1}ts = t^2.$$

Every normal 2-subgroup of such a quotient is central and lies in the unramified cyclic direction.

Proof. Conjugation preserves order, whereas $|t^2| = |t|/\gcd(2, |t|)$, so $|t|$ is odd. Let $I = \langle t \rangle$ and suppose sI has order n . Then $s^n = t^m$. Since s^n commutes with s , the relation $t^s = t^2$ gives $t^m = t^{2m}$, hence $m \equiv 0 \pmod{|t|}$. Thus the extension splits.

If N is a normal 2-subgroup, then $N \cap I = 1$ and $[N, I] \leq N \cap I = 1$. Hence N centralizes I . It is therefore contained in the 2-primary part of the kernel of the Frobenius action on I , which is central and unramified. $\square$

The first comparison between the two sources happens on tame quotients, where both collapse onto the same explicit group.

Proposition 3.2 (Common tame quotient). *There are canonical isomorphisms*

$$\Gamma_A/W_A \cong T_{\text{tame}} \cong G_{\mathbb{Q}_2}/W_F,$$

where W_F is wild inertia.

Proof. In the quotient by W_A , the elements x_0, x_1 vanish. By Lemma 3.1, τ has odd order in every finite quotient, so $\tau_2^\omega = 1$. Hence $u_i = d_0 = z_0 = c_0 = d_g = h_c = h_0 = 1$; the words σ_2 and g_0 may remain nontrivial, but they only conjugate the identity. The wild relation therefore disappears. For the local group, the tame quotient has pro-prime-to-2 cyclic inertia and procyclic unramified quotient; geometric Frobenius acts on tame inertia by $x \mapsto x^2$. With $x^g = g^{-1}xg$, this is precisely the relation $\tau^\sigma = \tau^2$. $\square$

The description of the tame quotient is Iwasawa's; see [1, Theorem (7.5.3)] and [7, Chapter IV].

Lemma 3.3 (The wild subgroup is the characteristic 2-core). *The tame group T_{tame} has no nontrivial closed normal pro-2 subgroup. Consequently*

$$O_2(\Gamma_A) = W_A, \quad O_2(G_{\mathbb{Q}_2}) = W_F.$$

In particular, the marked wild subgroups of the two sources are characteristic.

Proof. Let $I = \overline{\langle \tau \rangle} \triangleleft T_{\text{tame}}$ be tame inertia; it is pro-odd. If $N \triangleleft T_{\text{tame}}$ is pro-2, then

$$N \cap I = 1, \quad [N, I] \leq N \cap I = 1,$$

so N centralizes I . Its image in $T_{\text{tame}}/I \cong \widehat{\mathbb{Z}}$ lies in the 2-primary factor and has a 2-adic geometric Frobenius coordinate $a \in \mathbb{Z}_2$. For every $m \geq 1$, use the finite inertia quotient of order

$$e_m = 2^{2^m} - 1.$$

Centrality forces $2^a \equiv 1 \pmod{e_m}$. The order of 2 modulo e_m is exactly 2^m , and hence $a \in 2^m \mathbb{Z}_2$ for every m . Thus $a = 0$, so $N \leq I$, and therefore $N = 1$.

The subgroup W_A is normal pro-2 and $\Gamma_A/W_A \cong T_{\text{tame}}$. Every normal pro-2 subgroup of Γ_A maps trivially to T_{tame} and is contained in W_A . This proves $O_2(\Gamma_A) = W_A$. The identical argument applied to $1 \rightarrow W_F \rightarrow G_{\mathbb{Q}_2} \rightarrow T_{\text{tame}} \rightarrow 1$ gives the local assertion. $\square$

3.1 The standard rank-three dyadic Demushkin group

Labute's classification identifies $G_{\mathbb{Q}_2}(2)$ with the group below as an abstract pro-2 group, but it does not choose an isomorphism preserving the full unramified character. An abstract identification is insufficient because the tame and pro-2 quotients must agree over their common quotient $\mathbb{Z}_2$. The canonical Demushkin orientation supplies the additional rigidity: together with the unramified character, it separates the three summands of the abelianization.

Put

$$D_0 = \langle A, S, Y \mid A^2 S^4 [S, Y] = 1 \rangle_{\text{pro-2}} \quad (3.1)$$

and

$$B_{\text{ab}} = D_0^{\text{ab}} = \langle \bar{A}, \bar{S}, \bar{Y} \mid 2\bar{A} + 4\bar{S} = 0 \rangle_{\mathbb{Z}_2}. \quad (3.2)$$

Let

$$t = \bar{A} + 2\bar{S}, \quad \eta = (-3)^{-1} \in \mathbb{Z}_2^\times. \quad (3.3)$$

Then

$$B_{\text{ab}} = C_2 t \oplus \mathbb{Z}_2 \bar{S} \oplus \mathbb{Z}_2 \bar{Y}. \quad (3.4)$$

Lemma 3.4 (Standard dyadic group and its orientation). *The group D_0 is the standard rank-three, $q = 2$ Demushkin group, and $G_{\mathbb{Q}_2}(2)$ is abstractly isomorphic to D_0 . The canonical Demushkin orientation of D_0 factors through B_{ab} and is the character*

$$\chi_0(t) = -1, \quad \chi_0(\bar{S}) = 1, \quad \chi_0(\bar{Y}) = \eta. \quad (3.5)$$

Equivalently,

$$\chi_0(\bar{A}) = -1, \quad \chi_0(\bar{S}) = 1, \quad \chi_0(\bar{Y}) = (-3)^{-1}.$$

Proof. The presentation (3.1) is the odd-rank dyadic case of Labute's classification, with parameter $q = 2$ and secondary exponent 4; see [2, Theorem 8]. The canonical character for the odd-rank normal form is computed in [2, Theorem 4, case (2)]: for the relation $x_1^2 x_2^{2^f} [x_2, x_3]$ it takes the values $-1, 1, (1 - 2^f)^{-1}$ on x_1, x_2, x_3 . With $f = 2$ and the standing commutator convention, this gives $-1, 1, (1 - 4)^{-1} = (-3)^{-1}$ on A, S, Y , respectively. The maximal pro-2 Galois group of $\mathbb{Q}_2$ is the rank-three local Demushkin group with these parameters, so the abstract isomorphism follows from the same classification. $\square$

Lemma 3.5 (Marked abelianization, orientation, and initial form). *Use arithmetic local reciprocity rec , normalized so that $\text{rec}(2)$ is arithmetic Frobenius, and normalize ν_{ur} geometrically. Put*

$$\bar{s} = \text{rec}(2)^{-1} = \text{rec}(1/2), \quad \bar{\epsilon} = \text{rec}(-1), \quad \bar{y} = \text{rec}(-3), \quad \bar{a} = \bar{\epsilon}\bar{s}^{-2} = \text{rec}(-4).$$

Then the pro-2 abelianization of $D_{\text{loc}} = G_{\mathbb{Q}_2}(2)$ is identified with B_{ab} by

$$\bar{A} \longmapsto \bar{a}, \quad \bar{S} \longmapsto \bar{s}, \quad \bar{Y} \longmapsto \bar{y}.$$

Under this identification,

$$\nu_{\text{ur}}(\bar{a}, \bar{s}, \bar{y}) = (-2, 1, 0), \quad \chi_{\text{loc}}(\bar{a}, \bar{s}, \bar{y}) = (-1, 1, (-3)^{-1}), \quad (3.6)$$

where χ_{loc} is the canonical Demushkin orientation. In the basis of $H^1(D_{\text{loc}}, \mathbb{F}_2)$ dual to $(\bar{a}, \bar{s}, \bar{y})$, the Demushkin quadratic initial form is

$$\alpha^2 + \beta\gamma + \gamma\beta.$$

Consequently a Demushkin relator for lifts of these classes has the same quadratic initial form as

$$r_0 = A^2 S^4 [S, Y],$$

namely $A^2 + [S, Y]$ in degree two. Moreover the pair

$$(\nu_{\text{ur}}, \chi_{\text{loc}}) : B_{\text{ab}} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2^\times$$

is injective.

Proof. We compute successively the unramified character, the cyclotomic orientation, the quadratic initial form, and the injectivity of the two-character marking.

Since ν_{ur} is normalized by geometric Frobenius while $\text{rec}(2)$ is arithmetic Frobenius, one has $\nu_{\text{ur}}(\text{rec}(2)) = -1$. Hence

$$\nu_{\text{ur}}(\bar{s}) = 1, \quad \nu_{\text{ur}}(\bar{y}) = 0, \quad \nu_{\text{ur}}(\bar{a}) = -2.$$

The pro-2 completion of $\mathbb{Q}_2^\times$ is

$$\widehat{\mathbb{Q}_2^\times}^{(2)} \cong \langle 2 \rangle_{\mathbb{Z}_2} \oplus \langle -1 \rangle_{C_2} \oplus \langle -3 \rangle_{\mathbb{Z}_2}.$$

The single relation among $\bar{A}, \bar{S}, \bar{Y}$ is

$$\bar{a}^2 \bar{s}^4 = \text{rec}((-4)^2 (1/2)^4) = \text{rec}(1) = 1,$$

that is, additively, $2\bar{A} + 4\bar{S} = 0$.

For a local Demushkin group the canonical orientation is the 2-adic cyclotomic character. With the reciprocity convention above, local reciprocity gives

$$\chi_{\text{loc}}(\text{rec}(u)) = u^{-1} \quad (u \in \mathbb{Z}_2^\times),$$

while the uniformizer component acts trivially on the totally ramified extension $\mathbb{Q}_2(\mu_{2^\infty})/\mathbb{Q}_2$. This proves the orientation row in (3.6).

Modulo squares, the classes are

$$\bar{a} \equiv [-1], \quad \bar{s} \equiv [2], \quad \bar{y} \equiv [-3].$$

For odd units u, v , write

$$\epsilon(u) = \frac{u-1}{2} \bmod 2, \quad \omega(u) = \frac{u^2-1}{8} \bmod 2.$$

The dyadic Hilbert-symbol formula

$$(2^\alpha u, 2^\beta v)_2 = (-1)^{\epsilon(u)\epsilon(v) + \alpha\omega(v) + \beta\omega(u)}$$

gives, in the square-class basis $(-1, 2, -3)$,

$$(-1, -1)_2 = -1, \quad (2, -3)_2 = -1,$$

with all other pairings among these basis elements trivial, up to symmetry. Under Kummer theory, cup product is the Hilbert symbol. Thus the dual basis has Bockstein/cup initial form $\alpha^2 + \beta\gamma + \gamma\beta$, which is exactly the degree-two initial form of $A^2S^4[S, Y]$.

Finally, in the decomposition (3.4),

$$\nu_{\text{ur}}(t, \bar{S}, \bar{Y}) = (0, 1, 0), \quad \chi_{\text{loc}}(t, \bar{S}, \bar{Y}) = (-1, 1, \eta).$$

The element η topologically generates $1 + 4\mathbb{Z}_2$, because $v_2(\eta - 1) = 2$. The two displayed characters therefore detect the three direct summands of (3.4), proving injectivity. $\square$

The Demushkin classification underlying this normalization goes back to [3] and [2]; the local reciprocity inputs are standard, see [1, Chapter VII §7.1] and [7, Chapters XI–XIII].

Remark 3.6 (Why the orientation cannot be omitted). An un-oriented marked-relator normalization with only the abelianized relation and quadratic initial form is false. Let $F = \langle A, S, Y \rangle_{\text{pro-}2}$ and define

$$\psi(A) = AY^{-4}, \quad \psi(S) = SY^2, \quad \psi(Y) = Y.$$

This is an automorphism, with inverse $A \mapsto AY^4$, $S \mapsto SY^{-2}$, $Y \mapsto Y$. For $r_0 = A^2S^4[S, Y]$, put $r = \psi^{-1}(r_0)$. On the free abelianization,

$$\psi_{\text{ab}}(2\bar{A} + 4\bar{S}) = 2(\bar{A} - 4\bar{Y}) + 4(\bar{S} + 2\bar{Y}) = 2\bar{A} + 4\bar{S}.$$

Moreover

$$r = (AY^4)^2(SY^{-2})^4[SY^{-2}, Y],$$

so, modulo the third Zassenhaus term, its initial form is $A^2 + [S, Y]$. Thus r has the same abelianized relation and quadratic initial form as r_0 , and its quotient is abstractly D_0 . However, under the isomorphism induced by ψ , the transported canonical orientation takes S to $\eta^2 \neq 1$. Therefore no isomorphism from this quotient to D_0 which is the identity on B_{ab} can exist. The orientation-preserving lifting statement Proposition 3.9, rather than the false un-oriented normalization, is the exact structural input needed below.

3.2 Automorphisms preserving the orientation

To correct the action of an abstract isomorphism on B_{ab} , it is enough to show that every automorphism of B_{ab} preserving χ_0 is induced by an automorphism of D_0 .

Lemma 3.7 (Cyclotomic powering of the three peripheral classes). *Let*

$$\Delta = \langle P, T \rangle_{\text{pro-2}}, \quad C = (PT)^{-1}.$$

For every $u \in \mathbb{Z}_2^\times$ there are an automorphism $\varphi_u \in \text{Aut}(\Delta)$ and elements $c_P, c_T, c_C \in \Delta$ such that

$$\varphi_u(P) = c_P^{-1} P^u c_P, \quad \varphi_u(T) = c_T^{-1} T^u c_T, \quad \varphi_u(C) = c_C^{-1} C^u c_C. \quad (3.7)$$

Proof. We obtain the required outer automorphism from the arithmetic Galois action and then choose a representative that realizes the three peripheral conjugacy classes simultaneously.

Identify Δ with the maximal pro-2 geometric fundamental group of

$$X = \mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\},$$

with P, T, C the three compatible peripheral inertia generators. The Galois action on the geometric fundamental group gives

$$G_{\mathbb{Q}} \longrightarrow \text{Out}(\Delta).$$

The 2-adic cyclotomic character is surjective, since

$$\text{Gal}(\mathbb{Q}(\mu_{2^\infty})/\mathbb{Q}) \cong \mathbb{Z}_2^\times.$$

Each of the three cusps is rational, so its peripheral conjugacy class is preserved. The corresponding decomposition group acts on the procyclic inertia group through the cyclotomic character; see [8, Section 3.3 and Definition 37]. Concretely, if z is a local parameter and a chosen inertia generator sends $z^{1/2^n}$ to $\zeta_{2^n} z^{1/2^n}$, then the arithmetic action of g replaces ζ_{2^n} by $\zeta_{2^n}^{\chi_2(g)}$. With the opposite convention for representing the outer action, the exponent is $\chi_2(g)^{-1}$. Accordingly choose g with $\chi_2(g) = u$ in the first convention and with $\chi_2(g) = u^{-1}$ in the second. In either convention the resulting outer automorphism sends each peripheral generator to a conjugate of its u -th power. Choosing a representative of the outer class gives (3.7). $\square$

Lemma 3.8 (Square-root and HNN lifting). *For every $u \in \mathbb{Z}_2^\times$ there is an automorphism $\Psi_u \in \text{Aut}(D_0)$ such that, on B_{ab} ,*

$$\bar{A} \longmapsto u\bar{A}, \quad \bar{S} \longmapsto u\bar{S}. \quad (3.8)$$

Proof. The proof first lifts peripheral powering across the square-root relation, then transports the lift to the free pro-2 group on S, A , and finally extends it over the HNN stable letter Y .

Start with φ_u and the conjugators from Lemma 3.7. Consider

$$E_{\square} = \langle P, T, A \mid PTA^2 = 1 \rangle_{\text{pro-2}} \cong \langle P, A \rangle_{\text{pro-2}}, \quad C = A^2.$$

View the words c_P, c_T, c_C in $E_{\square}$ via P, T , and define

$$P \longmapsto \varphi_u(P), \quad T \longmapsto \varphi_u(T), \quad A \longmapsto c_C^{-1} A^u c_C.$$

The defining relation is preserved, because

$$(c_C^{-1}A^u c_C)^2 = c_C^{-1}C^u c_C = \varphi_u(C) = (\varphi_u(P)\varphi_u(T))^{-1}.$$

Modulo the Frattini subgroup, the free basis P, A is sent to $u\bar{P}, u\bar{A}$, hence to $\bar{P}, \bar{A}$. The pro-2 Burnside basis theorem and Hopficity therefore show that this endomorphism is an automorphism; call it θ_u .

In the free pro-2 group $F_{S,A} = \langle S, A \rangle_{\text{pro-2}}$ put

$$P = S^3, \quad T = S^{-3}A^{-2}.$$

Since $3 \in \mathbb{Z}_2^\times$, the pair P, A is again a free basis of $F_{S,A}$, and $PTA^2 = 1$. For every odd integer m , the power map $x \mapsto x^m$ is bijective on a pro-2 group, with inverse $x \mapsto x^{m^{-1}}$ for $m^{-1} \in \mathbb{Z}_2^\times$. Comparing the unique cube roots of the two sides of (3.7) therefore gives

$$\theta_u(S) = c_P^{-1}S^u c_P, \quad \theta_u(T) = c_T^{-1}T^u c_T.$$

The relation of D_0 is equivalently

$$Y^{-1}SY = T, \quad T = S^{-3}A^{-2}. \quad (3.9)$$

Define an endomorphism of the presentation (3.9) by

$$\Psi_u|_E = \theta_u, \quad \Psi_u(Y) = c_P^{-1}Y c_T.$$

Then

$$\Psi_u(Y)^{-1}\Psi_u(S)\Psi_u(Y) = c_T^{-1}Y^{-1}S^u Y c_T = c_T^{-1}T^u c_T = \Psi_u(T),$$

so the relation is respected. On the Frattini quotient of D_0 , the images of A, S, Y form a triangular basis with diagonal entries 1, 1, 1. Consequently Ψ_u is surjective and hence an automorphism. Its action on $\bar{A}, \bar{S}$ is (3.8). $\square$

Orientation-preserving automorphisms of the abelianization must now be lifted to the full pro-2 model.

Proposition 3.9 (Lifting the orientation-preserving abelian automorphisms). *The natural map*

$$\text{Aut}(D_0) \longrightarrow \text{Aut}(B_{\text{ab}}, \chi_0) \quad (3.10)$$

is surjective. More explicitly, every element of $\text{Aut}(B_{\text{ab}}, \chi_0)$ has the unique form

$$\alpha_{u,b}: \quad t \longmapsto t, \quad \bar{S} \longmapsto u\bar{S}, \quad \bar{Y} \longmapsto \bar{Y} + b\bar{S}, \quad (3.11)$$

with $u \in \mathbb{Z}_2^\times$ and $b \in \mathbb{Z}_2$, and every $\alpha_{u,b}$ lifts to $\text{Aut}(D_0)$.

Proof. We first determine the orientation-preserving automorphisms of the abelianization, then realize their unit-scaling part, and finally realize the remaining shear.

Because $v_2(\eta - 1) = 2$, the element η topologically generates $1 + 4\mathbb{Z}_2$. From (3.4) and (3.5) one obtains

$$\ker \chi_0 = \mathbb{Z}_2 \bar{S}.$$

The torsion subgroup of B_{ab} is the order-two group generated by t . Hence an orientation-preserving automorphism fixes t , sends $\bar{S}$ to $u\bar{S}$ for a unique unit u , and sends $\bar{Y}$ to $\bar{Y} + b\bar{S}$ for a unique b . Conversely, every map (3.11) is an orientation-preserving automorphism of B_{ab} .

The automorphism Ψ_u of Lemma 3.8 sends $\bar{A}, \bar{S}$ to $u\bar{A}, u\bar{S}$, so it fixes $t = \bar{A} + 2\bar{S}$ and sends $\bar{S}$ to $u\bar{S}$. The canonical orientation is the action on the dualizing module and is therefore preserved by every automorphism of D_0 . Thus Ψ_u induces $\alpha_{u,b(u)}$ for some $b(u) \in \mathbb{Z}_2$.

For $b \in \mathbb{Z}_2$, define

$$\Theta_b(A) = A^{S^b}, \quad \Theta_b(S) = S, \quad \Theta_b(Y) = YS^b, \quad (3.12)$$

Here $A^{S^b} = (S^b)^{-1}AS^b$ is conjugation, not the product AS^b . This is an automorphism of the free pro-2 group on A, S, Y , with inverse Θ_{-b} . Moreover

$$[S, YS^b] = [S, Y]^{S^b},$$

and therefore

$$\Theta_b(A^2S^4[S, Y]) = (A^2)^{S^b}S^4[S, Y]^{S^b} = (A^2S^4[S, Y])^{S^b}.$$

It therefore descends to an automorphism of D_0 , inducing $\alpha_{1,b}$. The composite $\Theta_{b-b(u)} \circ \Psi_u$ induces $\alpha_{u,b}$, proving surjectivity. $\square$

3.3 Matching the unramified marking

The preceding surjectivity permits an arbitrary abstract isomorphism $G_{\mathbb{Q}_2}(2) \cong D_0$ to be corrected so that it induces the identity on the marked abelianization B_{ab} . The unramified character is then preserved integrally, not only after reduction modulo 2.

Proof of Proposition 1.1. By Lemma 3.4, choose an abstract isomorphism

$$f : D_{\text{loc}} \xrightarrow{\sim} D_0.$$

Use Lemma 3.5 to identify $D_{\text{loc}}^{\text{ab}}$ with B_{ab} , and use (3.2) for D_0^{ab} . Canonical orientations are functorial under group isomorphisms. The orientation row (3.6) and (3.5) therefore imply

$$\chi_0 \circ f_{\text{ab}} = \chi_{\text{loc}} = \chi_0.$$

Thus $f_{\text{ab}} \in \text{Aut}(B_{\text{ab}}, \chi_0)$. By Proposition 3.9, choose $\gamma \in \text{Aut}(D_0)$ with

$$\gamma_{\text{ab}} = f_{\text{ab}}^{-1}.$$

Then

$$\beta = \gamma \circ f : D_{\text{loc}} \xrightarrow{\sim} D_0$$

induces the identity on the marked abelianization B_{ab} .

Set

$$a = \beta^{-1}(A), \quad s = \beta^{-1}(S), \quad y = \beta^{-1}(Y).$$

These elements topologically generate D_{loc} , satisfy $a^2s^4[s, y] = 1$, and have the prescribed abelianized classes. Hence

$$\nu_{\text{ur}}(a, s, y) = (-2, 1, 0),$$

which proves Proposition 1.1. $\square$

3.4 The maximal pro-2 quotient of the candidate

Proposition 3.10 (Maximal pro-2 quotient). *The maximal pro-2 quotient of Γ_A is*

$$\Pi = \left\langle \sigma, x_0, x_1 \mid x_0^{\sigma^2} x_0 [x_1, \sigma] = 1 \right\rangle_{\text{pro-2}}. \quad (3.13)$$

The group Π is abstractly isomorphic to $G_{\mathbb{Q}_2}(2)$. Equipped with

$$\nu_2 : \Pi \twoheadrightarrow \mathbb{Z}_2, \quad \nu_2(\sigma) = 1, \quad \nu_2(x_0) = \nu_2(x_1) = 0, \quad (3.14)$$

it is isomorphic to the fully unramified marked pair $(G_{\mathbb{Q}_2}(2), \nu_{\text{ur}})$.

Proof. In a finite 2-group quotient, [Lemma 3.1](#) forces $\tau = 1$, and ω_2 acts as the identity. Hence

$$u_i = x_i, \quad d_0 = 1, \quad c_0 = 1, \quad g_0 = \sigma^2, \quad d_g = h_c = 1,$$

and

$$h_0 = x_0^{\sigma^2} x_0.$$

Moreover

$$u_1^{-1} x_1^{\sigma} = x_1^{-1} x_1^{\sigma} = [x_1, \sigma].$$

Thus [\(1.6\)](#) becomes precisely the relator in [\(3.13\)](#). Conversely, every finite quotient of [\(3.13\)](#) is admissible after setting $\tau = 1$. The fully marked local identification is given by [Corollary 3.12](#). $\square$

Proposition 3.11 (Nielsen transform to the pro-2 boundary). *For any pro-2 group with topological generators a, s, y and unique defining relation*

$$a^2 s^4 [s, y] = 1, \quad (3.15)$$

the Nielsen change

$$\sigma = s, \quad x_1 = y, \quad x_0 = s^{-2} a^{-1} \quad (3.16)$$

changes the closed normal closure of the relator to that of

$$x_0^{\sigma^2} x_0 [x_1, \sigma]. \quad (3.17)$$

Consequently [Proposition 1.1](#) identifies $D = G_{\mathbb{Q}_2}(2)$ abstractly with Π .

Proof. Under [\(3.16\)](#), one has $a = x_0^{-1} \sigma^{-2}$ and therefore

$$a^2 s^4 = x_0^{-1} (x_0^{-1})^{\sigma^2}.$$

Writing $P = x_0^{\sigma^2} x_0$ and $Q = [x_1, \sigma]$, the inverse of the standard relator is QP . The words QP and PQ are cyclic conjugates, so they have the same closed normal closure. This proves the assertion. $\square$

Corollary 3.12 (Fully marked form). *The generators furnished by [Proposition 1.1](#) and transformed by [\(3.16\)](#) satisfy*

$$\nu_{\text{ur}}(\sigma) = 1, \quad \nu_{\text{ur}}(x_0) = \nu_{\text{ur}}(x_1) = 0,$$

and [\(3.17\)](#) is the unique defining relation. Hence $(D, \nu_{\text{ur}}) \cong (\Pi, \nu_2)$.

Proof. The relation is [Proposition 3.11](#). The values follow directly from (1.4):

$$\nu_{\text{ur}}(x_0) = -2\nu_{\text{ur}}(s) - \nu_{\text{ur}}(a) = 0.$$

□

Remark 3.13 (The integral marking is not detected by the mod-2 cup form). The more familiar unmarked Labute relator

$$x_0^2 x_1^4 [x_1, \sigma]$$

has the same quadratic initial form $b^2 + ac + ca$ as (3.17). It does not, however, admit the marking

$$\nu(\sigma) = 1, \quad \nu(x_0) = \nu(x_1) = 0$$

required by the tame boundary. In those Labute variables the canonical orientation has values

$$\chi(x_0) = -1, \quad \chi(x_1) = 1, \quad \chi(\sigma) = (-3)^{-1}.$$

The abelianization is $\mathbb{Z}_2^2 \oplus C_2$, and $\bar{x}_1$ has infinite order. The asserted marking would nevertheless put x_1 in $\ker \nu \cap \ker \chi$, contradicting the injectivity of (ν_{ur}, χ) on D^{ab} . Thus agreement of the maximal pro-2 quotient as an unmarked Demushkin group is insufficient. The Nielsen transformation (3.16) replaces that relator by (3.17). It has the same mod-2 two-jet; its full unramified marking follows from [Proposition 1.1](#).

For the boundary construction it is not enough that the tame and pro-2 quotients exist separately: they must carry one compatible unramified character. We arrange this next.

Proposition 3.14 (Fully marked tame and pro-2 quotients). *Let*

$$\nu_t : T_{\text{tame}} \twoheadrightarrow \mathbb{Z}_2, \quad \nu_t(\sigma) = 1, \quad \nu_t(\tau) = 0,$$

and equip Π with (3.14). For each $\Gamma \in \{\Gamma_A, G_{\mathbb{Q}_2}\}$, the tame quotient map and the maximal pro-2 quotient map may be chosen so that their composites with ν_t and ν_2 are the same natural unramified character

$$\nu_\Gamma : \Gamma \twoheadrightarrow \mathbb{Z}_2.$$

The compatibility therefore holds after passage to every finite quotient C_{2^m} , not only modulo 2.

Proof. For the candidate, the assignment

$$\sigma \longmapsto 1, \quad \tau, x_0, x_1 \longmapsto 0$$

respects both defining relations and induces the displayed maps on the tame and pro-2 quotients. For the local source, use [Corollary 3.12](#). The tame geometric Frobenius and the pro-2 generator σ have the same full unramified image, while x_0, x_1 lie in its kernel. □

Remark 3.15 (Why the full marking is load-bearing). The construction

$$\partial_{\text{bd}} = T_{\text{tame}} \times_{\mathbb{Z}_2} \Pi$$

uses the full marked isomorphism $(\Pi, \nu_2) \cong (G_{\mathbb{Q}_2}(2), \nu_{\text{ur}})$. An unmarked Demushkin classification identifies only the abstract pro-2 group; it does not fix the terminal comparison after passage to all finite quotients C_{2^m} of the unramified coordinate.

Let (a, b, c) be the basis of $H^1(\Pi, \mathbb{F}_2)$ dual to (σ, x_0, x_1) . The quadratic initial form of the relator in (3.13) is

$$b^2 + ac + ca.$$

Consequently the scalar cup/Bockstein table is

$$\mathcal{H}((a, b, c), (a', b', c')) = ac' + ca' + bb'. \quad (3.18)$$

This is the local Hilbert pairing on $\mathbb{Q}_2^\times / \mathbb{Q}_2^{\times 2}$ under the corresponding basis.

4 The common tame–pro-2 boundary and the global induction

We combine the compatible tame and maximal pro-2 quotient maps into a common boundary over the full unramified quotient. This boundary lets us formulate the fixed-frame exact-image equality that the subsequent deformation, quadratic, Fourier, and induction arguments will prove. The compatible quotient maps obtained in Section 3 are assembled here into a single boundary condition. This produces the epimorphisms $b_\Gamma : \Gamma \twoheadrightarrow \partial_{\text{bd}}$ and the finite targets on which the induction of Section 9 is carried out.

The tame quotient and the maximal pro-2 quotient share the full unramified pro-2 quotient:

$$T_{\text{tame}} \xrightarrow{\nu_t} \mathbb{Z}_2 \xleftarrow{\nu_2} \Pi, \quad \partial_{\text{bd}} = T_{\text{tame}} \times_{\mathbb{Z}_2} \Pi. \quad (4.1)$$

The fibre product records the requirement that the two quotient maps have the same unramified coordinate. Thus their markings cannot be chosen independently, or merely matched after reduction modulo 2. For $\Gamma \in \{\Gamma_A, G_{\mathbb{Q}_2}\}$, the compatible tame and maximal pro-2 maps of Proposition 3.14 combine to an epimorphism

$$b_\Gamma : \Gamma \twoheadrightarrow \partial_{\text{bd}}. \quad (4.2)$$

Surjectivity follows from profinite Goursat. Any common quotient of T_{tame} and the pro-2 group Π is a pro-2 quotient of T_{tame} , hence a quotient of its maximal pro-2 quotient $\mathbb{Z}_2$; Proposition 3.14 shows that the full $\mathbb{Z}_2$ quotient occurs.

Fix a finite tame quotient H , an epimorphism $\alpha : T_{\text{tame}} \twoheadrightarrow H$, an elementary abelian 2-group E , and a homomorphism $\bar{\psi} : \Pi \rightarrow E$. They define

$$\beta : \partial_{\text{bd}} \longrightarrow H \times E, \quad \beta(t, p) = (\alpha(t), \bar{\psi}(p)). \quad (4.3)$$

We compare only surjections that induce the fixed map to H and the fixed elementary decoration in E . A finite target carrying these two prescribed pieces of data will be called *boundary-framed*.

Definition 4.1 (Boundary-framed marked target). A *boundary-framed marked target* is a tuple

$$\mathcal{Y} = (Y, L_Y, \pi_Y, \theta_Y)$$

where $L_Y \triangleleft Y$ is a finite 2-group, $\pi_Y : Y \twoheadrightarrow H$ has kernel L_Y , and $\theta_Y : Y \rightarrow E$ is a homomorphism. Put

$$q_Y = (\pi_Y, \theta_Y) : Y \longrightarrow H \times E.$$

For $\Gamma \in \{\Gamma_A, G_{\mathbb{Q}_2}\}$ define

$$e_\Gamma^\beta(\mathcal{Y}) = \#\{f : \Gamma \twoheadrightarrow Y : q_Y f = \beta b_\Gamma\}. \quad (4.4)$$

The tuple $\mathcal{Y}$ therefore fixes, on the same finite target, both the tame quotient $Y/L_Y \cong H$ and the homomorphism $Y \rightarrow E$. The equation $q_Y f = \beta b_\Gamma$ counts surjections inducing both prescribed maps. Since the wild subgroup is the kernel of the tame quotient, it also forces W_A or W_F into L_Y .

An *exact-image stratum* consists of maps whose image is a specified subgroup J , rather than maps that merely land in J . If $J \leq Y$ projects onto H , then

$$\mathcal{J} = (J, J \cap L_Y, \pi_Y|_J, \theta_Y|_J)$$

inherits the same structure. Consequently, partitioning maps according to their image subgroup produces targets of the same form.

Theorem 4.2 (Boundary-framed exact-image theorem). *For every boundary-framed marked target $\mathcal{Y}$,*

$$e_{\Gamma_A}^\beta(\mathcal{Y}) = e_{G_{\mathbb{Q}_2}}^\beta(\mathcal{Y}).$$

The same equality holds for every exact-image target $\mathcal{J}$ defined above.

The proof is a global induction over all boundary-framed marked targets, ordered by $|L_Y|$. Auxiliary central covers need not lie inside the original ambient group, which is why the induction is stated globally.

5 Linear lifting theory and duality

This section starts with a lower map and an elementary kernel and extracts the linear information needed by the target induction. The candidate and local cochain theories remain distinct; their common numerical and duality content is stated in [Corollary 5.17](#).

Fix a finite group C , a boundary-compatible epimorphism $\rho : \Gamma \twoheadrightarrow C$, and a finite elementary $\mathbb{F}_2[C]$ -module A . Here *elementary* means that the additive group of A is a finite-dimensional $\mathbb{F}_2$ -vector space; no semisimplicity assumption is made. A lifting datum over ρ is an extension

$$1 \longrightarrow A \longrightarrow \tilde{C} \xrightarrow{\pi} C \longrightarrow 1$$

inducing the prescribed C -action on A . In either source theory it has a degree-two obstruction

$$o_{\Gamma, \rho}(\tilde{C}) \in H_{\Gamma, \rho}^2(A).$$

The obstruction vanishes exactly when ρ lifts to $\tilde{C}$; after one lift is fixed, every other lift is obtained uniquely by multiplication by an element of $Z_{\Gamma, \rho}^1(A)$. Thus a nonempty lift set is a principal homogeneous space under $Z_{\Gamma, \rho}^1(A)$. Only the obstruction dimensions, torsor cardinalities, perfect pairings, and connecting homomorphisms will be compared; no map between the two cochain complexes or their degree-one cohomology spaces is used.

For $\Gamma = \Gamma_A$, varying the four distinguished generators through an extension by A gives the finite word complex

$$C_{\rho, A}^\bullet : 0 \longrightarrow A \xrightarrow{d_{\rho, A}^0} A^4 \xrightarrow{d_{\rho, A}^1} A^2 \longrightarrow 0. \quad (5.1)$$

Write S, T, X_0, X_1 for the actions of the four lower values on A . With the lift convention (a, g) in a semidirect product, simultaneous infinitesimal conjugation gives

$$d_{\rho, A}^0(v) = ((S - 1)v, (T - 1)v, (X_0 - 1)v, (X_1 - 1)v). \quad (5.2)$$

The map $d_{\rho,A}^1$ is the ordered pair of the exact tame- and wild-relator defects. Concretely, if the lower map has values in C and a lift to $C \ltimes A$ is written

$$(\rho(\sigma), a), \quad (\rho(\tau), b), \quad (\rho(x_0), c), \quad (\rho(x_1), d),$$

then $d_{\rho,A}^1(a, b, c, d)$ is the coefficient of A in the evaluated tame and wild relators. This construction is functorial in the C -module A because it is obtained by evaluating the same two finite words in semidirect products. Its degree-one kernel is denoted $Z_{A,\rho}^1(A)$ and its cohomology by $H_{A,\rho}^i(A)$.

The pro-2 normal-closure condition contributes no additional linear equation. The elementary kernel A lies in the marked normal 2-subgroup of the target, so multiplying any of the four chosen lifts by an element of A preserves the marked condition; the only defects are the two displayed relators.

For $\Gamma = G_{\mathbb{Q}_2}$, put

$$H_{F,\rho}^i(A) = H_{\text{cont}}^i(G_{\mathbb{Q}_2}, A_\rho), \quad Z_{F,\rho}^1(A) = Z_{\text{cont}}^1(G_{\mathbb{Q}_2}, A_\rho),$$

where the action is induced by ρ . These are the ordinary continuous Galois cohomology groups. No two-relator complex is assigned to $G_{\mathbb{Q}_2}$.

5.1 The evaluated Fox Jacobian

Let $\mathbf{S}, \mathbf{T}$ denote the actions of $\rho(\sigma), \rho(\tau)$ on A . If $e = |\mathbf{T}|$ is odd, put

$$P = 1 + \mathbf{T} + \cdots + \mathbf{T}^{e-1}.$$

The following calculation is valid for any finite coefficient module after replacing ω_2 by an integer representative modulo the exponent of the finite semidirect products used in the calculation.

Lemma 5.1 (Finite-exponent independence). *Suppose ν and ν' represent ω_2 modulo the exponent of every finite semidirect and Heisenberg group occurring in a fixed first- or second-order calculation. Replacing every ω_2 -power in the relators by ν or by ν' gives the same first derivative and the same mixed second derivative.*

Proof. The two ordinary powers define the same element in each finite group in which the relator is evaluated. First and mixed derivatives are, by definition, the noncentral and central coordinates of those finite evaluations. They are therefore identical. The representative may be chosen separately for each finite coefficient calculation; no global coherent integer is required. $\square$

Lemma 5.2 (Exact class-two identity). *Let*

$$1 \longrightarrow Z \longrightarrow \tilde{V} \longrightarrow V \longrightarrow 1$$

be a central extension in which V and Z are elementary abelian 2-groups, and let ϕ be an automorphism of $\tilde{V}$ fixing Z . For $X, D \in \tilde{V}$, put

$$\mathfrak{h}_\phi(X, D) = \phi(X)X\phi(D)D D^2[\phi(D), D].$$

Then

$$\mathfrak{h}_\phi(X, D) = \phi(X)XD^{-1}\phi(D). \tag{5.3}$$

Consequently:

1. if D and X have the same image in V , then $\mathfrak{h}_\phi(X, D) = X^2$;
2. if ϕ is the identity, then $\mathfrak{h}_\phi(X, D) = X^2$ for every D .

Proof. The group $\tilde{V}$ has class at most two, every commutator has order at most two, and every element has fourth power one. Put $A = \phi(D)$, $B = D$, and $k = [A, B]$. With our commutator convention $AB = BAk$; since k is central of order at most two and B^2 is central,

$$ABB^2k = BAB^2 = B^3A = B^{-1}A.$$

This proves (5.3).

If $D = Xz$ with $z \in Z$, then $\phi(z) = z$ and $D^{-1}\phi(D) = X^{-1}\phi(X)$. Hence $\mathfrak{h}_\phi(X, D) = \phi(X)^2 = X^2$, because $X^2 \in Z$ and ϕ fixes Z . If $\phi = 1$, (5.3) gives the second assertion directly. $\square$

Lemma 5.3 (The elementary and class-two shadow of h_0). *Let V be either the trivial module or a nontrivial simple tame $\mathbb{F}_2[C]$ -module. For an arbitrary lower map, the marked wild subgroup acts trivially on every simple factor, so the evaluated simple-factor derivatives are the same as those of the induced tame lower map. At that tame map:*

1. the first derivative of h_0 is the first derivative of x_0^2 , namely zero;
2. on the normalized degree-one representatives supported on x_0 , the mixed Heisenberg and extraspecial central coordinates of h_0 are respectively those of x_0^2 .

Thus h_0 may replace x_0^2 in every linear, cup-product, and determinant calculation used below.

Proof. We first compute the linear variation of h_0 and then treat its class-two central coordinate separately in the cases $P = 0$ and $P = 1$.

Let $U = S_2^\omega$. If the first-order variations of τ and x_0 are b and c , then

$$Dd_0 = Pb + (P + 1)c, \quad Dx_0 = c.$$

The variation of g_0 affects only terms of order at least two, because the base values of x_0 and d_0 at the tame map are both the identity. If $P = 0$, then $Dd_0 = c = Dx_0$, and the four linear terms in $\mathfrak{h}_{g_0}(x_0, d_0)$ cancel. If $P = 1$, then $T = 1$ and the 2-primary part U acts trivially on a simple characteristic-2 module; again the four linear terms cancel. This proves (i).

For the normalized cohomology representatives used in the mixed and quadratic calculations, only x_0 varies. If $P = 0$, the two elements d_0 and x_0 have the same image c in V , so Lemma 5.2(i) applies. If $P = 1$, then $Dd_0 = 0$ and $T = 1$. The 2-primary operator U acts trivially on the simple quotient V . Its lift to a Heisenberg or extraspecial coefficient group is therefore a central automorphism. Such an automorphism has order at most 2, because it is determined by a homomorphism from the elementary quotient to the central involution. Hence $g_0 = \sigma_2^2$ acts trivially on the full coefficient group, and Lemma 5.2(ii) applies. For the trivial module, $T = U = 1$, so the second case applies without modification. Taking the mixed Heisenberg central kernel gives the square diagonal $\lambda(c)$; taking the extraspecial central kernel gives $q(c)$. This proves (ii). $\square$

Lemma 5.4 (Full finite-word ledger). *At a tame lower map, write first-order variations of (σ, τ, x_0, x_1) as (a, b, c, d) . Let S, T be the lower actions of σ, τ , put $U = S_2^\omega$, and let $P = 1 + T + \cdots + T^{e-1}$ for the odd order e of T . The raw finite Fox rules give*

$$D(u_i) = P(Dx_i + D\tau), \quad D(d_0) = Pb + (P + 1)c, \quad D(z_0) = U^{-1}c.$$

The first-order contributions of the wild factors are

factor	a	b	c	d
h_0	0	0	0	0
u_1^{-1}	0	P	0	P
x_1^σ	0	0	0	S^{-1}
$[d_0, z_0]$	0	0	0	0.

Thus the wild row is $L_w = Pb + (P + S^{-1})d$.

Write b_{cl} for the bilinear commutator pairing of the chosen class-two coefficient group. For the mixed Heisenberg or extraspecial calculation on the normalized cohomology representative supported on x_0 , the complete central ledger is

factor	V -coordinate	central coordinate
h_0	0	$\lambda(c)$ in $\mathcal{H}(V)$, $q(c)$ for κ_q^0
u_1^{-1}	0	0
x_1^σ	0	0
$d_0 = (x_0\tau)_2^\omega x_0^{-1}$	$(P+1)c$	*
$z_0 = x_0^{\sigma^2}$	$U^{-1}c$	*
$[d_0, z_0]$	0	$b_{\text{cl}}((P+1)c, U^{-1}c)$.

The symbols * are irrelevant because d_0 and z_0 enter the relator only through their commutator. The safety of replacing h_0 by x_0^2 in these ledgers is exactly [Lemma 5.3](#): the variation of $g_0 = \sigma_2^2$ starts in order at least two; when $P = 0$ one has $Dd_0 = Dx_0$, so [Lemma 5.2\(i\)](#) applies; when $P = 1$, the lift of U to the class-two coefficient group is a central automorphism of order at most two, so g_0 acts trivially and [Lemma 5.2\(ii\)](#) applies.

Differentiating the relator words in these ledgers linearizes the lifting problem; the resulting Jacobian can be evaluated on each simple tame module.

Lemma 5.5 (Evaluated Jacobian). *Write the lift variables on (σ, τ, x_0, x_1) as $(a, b, c, d) \in A^4$. On a simple tame module the actual relation differential is*

$$d_{\rho, A}^1 = \begin{pmatrix} S^{-1}(1 + T) & S^{-1} + 1 + T & 0 & 0 \\ 0 & P & 0 & P + S^{-1} \end{pmatrix}. \quad (5.4)$$

Equivalently,

$$L_t = S^{-1}(1 + T)a + (S^{-1} + 1 + T)b, \quad (5.5)$$

$$L_w = Pb + (P + S^{-1})d. \quad (5.6)$$

Postcomposing the tame defect coordinate by S gives the row-normalized expression

$$(1 + T)a + (1 + S(1 + T))b,$$

which may be used for rank calculations only. The word complex, the mixed derivative, and the endpoint maps below use the unnormalized row [\(5.5\)](#).

Proof. Use $(u, g)(v, h) = (u + gv, gh)$ in $A \rtimes C$. A direct calculation gives

$$D(\tau^\sigma \tau^{-2})(a, b) = S^{-1}(T - 1)a + (S^{-1} - (1 + T))b.$$

Over $\mathbb{F}_2$ this is exactly (5.5).

For an odd-order T -orbit, differentiation of a ω_2 -power gives the norm projector P . By Lemma 5.3, the word h_0 has the first-order value of x_0^2 , hence zero. The remaining wild factors give

	a	b	c	d
h_0	0	0	0	0
u_1^{-1}	0	P	0	P
x_1^σ	0	0	0	S^{-1}
$[d_0, z_0]$	0	0	0	0.

Their sum is (5.6). Without the simplicity assumption, the same finite Fox rules define the word complex; the displayed reduction is the simple-factor calculation used in the devissage. $\square$

Here and below, the duality inputs on the Galois side are local Tate duality and the Euler–Poincaré characteristic; see [1, Theorems (7.2.6) and (7.3.1)] and [15, Chapter I].

5.2 The Fox–Heisenberg cup calculation

For an elementary module A , set

$$\mathcal{H}(A) = A \times A^\vee \times \mathbb{F}_2,$$

with multiplication

$$(a, \lambda, z)(a', \lambda', z') = (a + a', \lambda + \lambda', z + z' + \lambda(a')).$$

The C -action is diagonal. If a finite word has evaluated coordinates

$$(\bar{u}, D_u, D_u^\vee, \beta_u),$$

then multiplication and inversion obey

$$D_{uv} = D_u + \bar{u}D_v, \quad D_{uv}^\vee = D_u^\vee + \bar{u}D_v^\vee, \quad (5.7)$$

$$\beta_{uv} = \beta_u + \beta_v + (D_u^\vee)(\bar{u}D_v), \quad \beta_{u^{-1}} = \beta_u + (\bar{u}^{-1}D_u^\vee)(\bar{u}^{-1}D_u). \quad (5.8)$$

These are the finite Fox recursion formulas. The Stokes identities below are proved directly by conjugation in the finite Heisenberg group; no cellular diagonal approximation is used. All identities in this subsection are identities of finite word evaluations before cohomology. In particular, coefficient functoriality and the later passage to nonsplit elementary modules are proved at the level of complexes and mapping cones; no semisimple decomposition of the C -module A is used.

Lemma 5.6 (Strict coefficient naturality). *For every C -module map $f : A \rightarrow A'$, the mixed derivatives satisfy*

$$\beta_{\rho, r}^{A'}(f_*a, \lambda') = \beta_{\rho, r}^A(a, f^\vee \lambda').$$

Proof. Evaluate the same finite relator in the mixed Heisenberg group

$$A \times A'^\vee \times \mathbb{F}_2, \quad (a, \lambda, z)(a', \lambda', z') = (a + a', \lambda + \lambda', z + z' + \lambda(f(a'))).$$

Its two natural images in $\mathcal{H}(A')$ and $\mathcal{H}(A)$ have the same central coordinate. This gives the identity before passage to cohomology. $\square$

Lemma 5.7 (Finite-word Stokes formula). *Let $F_n = \langle g_1, \dots, g_n \rangle$ be a free group, let $\rho : F_n \rightarrow C$ be a homomorphism to a finite group, and write $c_i = \rho(g_i)$. Let $r \in F_n$ be an ordinary finite word with $\rho(r) = 1$, and let $\epsilon_i(r) \in \mathbb{F}_2$ be the total exponent of g_i in r , reduced modulo 2.*

For $x = (x_i) \in A^n$ and $y = (y_i) \in (A^\vee)^n$, evaluate r after replacing g_i by $(x_i, y_i, 0; c_i) \in \mathcal{H}(A) \rtimes C$. Write the result as

$$(L_r^A(x), L_r^{A^\vee}(y), \beta_r(x, y); 1).$$

Then, for $a \in A$ and $\lambda \in A^\vee$,

$$\beta_r(d^0 a, y) = \langle a, L_r^{A^\vee}(y) \rangle + \sum_{i=1}^n \epsilon_i(r) y_i(c_i a), \quad (5.9)$$

$$\beta_r(x, d^0 \lambda) = \langle L_r^A(x), \lambda \rangle + \sum_{i=1}^n \epsilon_i(r) \lambda(x_i), \quad (5.10)$$

where

$$d^0 a = ((c_i - 1)a)_i, \quad d^0 \lambda = ((c_i - 1)\lambda)_i.$$

Consequently, for words $r_1, \dots, r_m$ and coefficients $t_j \in \mathbb{F}_2$ satisfying

$$\sum_{j=1}^m t_j \epsilon_i(r_j) = 0 \quad (1 \leq i \leq n), \quad (5.11)$$

the corresponding traced mixed coordinate satisfies the two Stokes identities without correction terms.

Proof. The two identities follow from conjugating by a primal and a dual coefficient, after which the total exponent vector records the central correction contributed by each generator.

Put

$$p_a = (a, 0, 0; 1), \quad p_\lambda = (0, \lambda, 0; 1), \quad \mathbf{z}(t) = (0, 0, t; 1).$$

A direct multiplication in $\mathcal{H}(A) \rtimes C$ gives

$$\begin{aligned} p_a^{-1}(0, y_i, 0; c_i) p_a &= ((c_i - 1)a, y_i, 0; c_i) \mathbf{z}(y_i(c_i a)), \\ p_\lambda^{-1}(x_i, 0, 0; c_i) p_\lambda &= (x_i, (c_i - 1)\lambda, 0; c_i) \mathbf{z}(\lambda(x_i)). \end{aligned}$$

The elements $\mathbf{z}(t)$ are central of order at most two. Therefore, on evaluating an ordinary word, the extra central factor attached to the i th generator occurs exactly $\epsilon_i(r)$ times.

If only the dual variables y_i are present, the value of r is $(0, L_r^{A^\vee}(y), 0; 1)$. Conjugating this value by p_a contributes the central coordinate $L_r^{A^\vee}(y)(a)$. Adding the generatorwise central corrections gives (5.9). Repeating this conjugation calculation with p_λ starts from $(L_r^A(x), 0, 0; 1)$ and gives (5.10). Condition (5.11) cancels the correction terms. Every step is functorial in A . $\square$

The finite-word identity acquires its useful form after tracing.

Proposition 5.8 (Traced finite-word Stokes identity). *Let β_t and β_w be the mixed central coordinates obtained from the tame and wild relators, and set*

$$\mathcal{B}_{\rho, A} = \beta_t + \beta_w.$$

Let L_t and L_w denote the two unnormalized first relation differentials. Then, before passage to cohomology,

$$\mathcal{B}_{\rho,A}(d_{\rho,A}^0 a, y) = \langle a, L_t^{A^\vee}(y) + L_w^{A^\vee}(y) \rangle, \quad (5.12)$$

$$\mathcal{B}_{\rho,A}(x, d_{\rho,A^\vee}^0 \lambda) = \langle L_t^A(x) + L_w^A(x), \lambda \rangle. \quad (5.13)$$

The identities are natural in A . They are assertions about the sum of the two relator coordinates; in general they are false for either relator separately.

Proof. We compute the two relators' mod-2 exponent vectors, apply the finite-word Stokes formula to their sum, and then invoke finite-exponent independence and coefficient naturality.

Choose an odd ordinary integer representative of ω_2 which is valid in every finite semidirect-Heisenberg group in the calculation, and replace the two profinite relators by the resulting ordinary words r_t and r_w . In the ordered generator list (σ, τ, x_0, x_1) their mod-2 exponent vectors are

$$\epsilon(r_t) = (0, 1, 0, 0), \quad \epsilon(r_w) = (0, 1, 0, 0).$$

Indeed, $r_t = \tau^\sigma \tau^{-2}$ has odd total τ -exponent. For $r_w = h_0 u_1^{-1} x_1^\sigma c_0$, the commutator c_0 has zero exponent vector. The word h_0 also has zero exponent vector: the two x_0 -terms cancel modulo 2, the two occurrences of d_0 cancel, d_0^2 has even exponent vector, and h_c is a commutator. Finally, u_1^{-1} contributes $x_1 + \tau$ and x_1^σ contributes x_1 . Thus the sum of the two exponent vectors is zero.

Apply Lemma 5.7 with trace coefficients $(1, 1)$. The correction terms in (5.9) cancel and give (5.12); the correction terms in (5.10) cancel and give (5.13). Independence of the chosen integer representative is Lemma 5.1, and strict coefficient naturality is Lemma 5.6. $\square$

Remark 5.9 (The trace is essential). There is a small explicit regression test. Take $C = \mathrm{GL}_2(\mathbb{F}_2)$ with

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad S^{-1}TS = T^2,$$

and let $A = \mathbb{F}_2^2$ be the natural module. For $a = e_2$ and the dual variation supported on τ with value $e_2^\vee$, the tame relator alone has mixed central value 0, whereas pairing its dual first derivative with a gives 1. The wild relator supplies the compensating value. Thus a relator-by-relator Stokes formula would assert $0 = 1$, while the traced formula above is exact.

For $x \in A^4$ and $y \in (A^\vee)^4$, let

$$\mathcal{B}_{\rho,A}(x, y) = \beta_{\rho,r_t}(x, y) + \beta_{\rho,r_w}(x, y)$$

be the sum of the two mixed central coordinates. The following proposition constructs the chain map used below, including its degree-zero and degree-two components.

Proposition 5.10 (The Fox–Heisenberg chain map). *Under the canonical identifications*

$$\mathrm{Hom}((A^\vee)^2, \mathbb{F}_2) \cong A^2, \quad \mathrm{Hom}((A^\vee)^4, \mathbb{F}_2) \cong A^4, \quad \mathrm{Hom}(A^\vee, \mathbb{F}_2) \cong A,$$

define

$$D_{\rho,A}^0(a) = (a, a), \quad (5.14)$$

$$\langle D_{\rho,A}^1(x), y \rangle = \mathcal{B}_{\rho,A}(x, y), \quad (5.15)$$

$$D_{\rho,A}^2(u_t, u_w) = u_t + u_w. \quad (5.16)$$

Then $D_{\rho,A} = (D^0, D^1, D^2)$ is a natural chain map

$$D_{\rho,A} : C_{\rho,A}^\bullet \longrightarrow \text{Hom}_{\mathbb{F}_2}(C_{\rho,A^\vee}^\bullet, \mathbb{F}_2)[-2]. \quad (5.17)$$

More explicitly,

$$D_{\rho,A}^1 d_{\rho,A}^0 = (d_{\rho,A^\vee}^1)^\vee D_{\rho,A}^0, \quad (5.18)$$

$$D_{\rho,A}^2 d_{\rho,A}^1 = (d_{\rho,A^\vee}^0)^\vee D_{\rho,A}^1. \quad (5.19)$$

Proof. The endpoint maps are induced by the scalar trace $\text{tr}(y_t, y_w) = y_t + y_w$. Equations (5.12) and (5.13), proved by the direct finite-word calculation of Lemma 5.7, are exactly (5.18) and (5.19), respectively. Strict naturality in A is Lemma 5.6, Proposition 5.8. Hence (5.17) is a natural chain map in all three degrees. $\square$

Lemma 5.11 (Exact cone dévissage). *For an elementary $\mathbb{F}_2[C]$ -module A , put*

$$K(A) = \text{Cone}(D_{\rho,A} : C_{\rho,A}^\bullet \longrightarrow \text{Hom}_{\mathbb{F}_2}(C_{\rho,A^\vee}^\bullet, \mathbb{F}_2)[-2]). \quad (5.20)$$

For every short exact sequence of finite $\mathbb{F}_2[C]$ -modules

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

there is a degreewise short exact sequence of complexes

$$0 \longrightarrow K(A') \longrightarrow K(A) \longrightarrow K(A'') \longrightarrow 0. \quad (5.21)$$

Consequently, if two of $K(A')$, $K(A)$, and $K(A'')$ are acyclic, then so is the third. In particular, acyclicity for all simple modules implies acyclicity for every elementary module, including nonsplit and nonsemisimple modules.

Proof. We verify exactness for the word complexes, then for their duals, and finally for the associated mapping cones.

The finite-word construction is additive and exact in the coefficient module, so the source complexes form a degreewise short exact sequence

$$0 \longrightarrow C_{\rho,A'}^\bullet \longrightarrow C_{\rho,A}^\bullet \longrightarrow C_{\rho,A''}^\bullet \longrightarrow 0. \quad (5.22)$$

Dualizing the coefficient sequence gives

$$0 \longrightarrow A''^\vee \longrightarrow A^\vee \longrightarrow A'^\vee \longrightarrow 0.$$

Applying the same finite-word complex and then $\text{Hom}_{\mathbb{F}_2}(-, \mathbb{F}_2)$ reverses this sequence once more, and yields

$$\begin{aligned} 0 \longrightarrow \text{Hom}_{\mathbb{F}_2}(C_{\rho,A'^\vee}^\bullet, \mathbb{F}_2)[-2] &\longrightarrow \text{Hom}_{\mathbb{F}_2}(C_{\rho,A^\vee}^\bullet, \mathbb{F}_2)[-2] \\ &\longrightarrow \text{Hom}_{\mathbb{F}_2}(C_{\rho,A''^\vee}^\bullet, \mathbb{F}_2)[-2] \longrightarrow 0. \end{aligned} \quad (5.23)$$

For compactness write

$$C^\dagger(A) = \text{Hom}_{\mathbb{F}_2}(C_{\rho,A^\vee}^\bullet, \mathbb{F}_2)[-2].$$

Strict coefficient naturality of the Heisenberg central coordinate gives the commutative diagram

$$\begin{array}{ccccc} C_{\rho,A'}^\bullet & \longrightarrow & C_{\rho,A}^\bullet & \longrightarrow & C_{\rho,A''}^\bullet \\ \downarrow D_{\rho,A'} & & \downarrow D_{\rho,A} & & \downarrow D_{\rho,A''} \\ C^\dagger(A') & \longrightarrow & C^\dagger(A) & \longrightarrow & C^\dagger(A''). \end{array}$$

Here the rows are the nonzero portions of (5.22) and (5.23), and commutativity is Lemma 5.6, Proposition 5.10. Mapping cones are formed degree by degree, so this diagram gives (5.21). The associated long exact cohomology sequence proves the two-out-of-three assertion. Applying it successively to a composition series of A proves the final statement. $\square$

Lemma 5.12 (Simple characteristic-two modules are tame). *Let V be a simple $\mathbb{F}_2[C]$ -module occurring in the induction. Then the marked normal 2-subgroup of C acts trivially on V . Hence V factors through the finite tame quotient of C .*

Proof. A finite 2-group acting on a nonzero vector space over $\mathbb{F}_2$ has a nonzero fixed vector. Since the 2-subgroup is normal in C , the fixed subspace is a nonzero C -submodule. Simplicity forces this fixed subspace to be all of V . $\square$

Lemma 5.13 (Normal forms for nontrivial simple modules). *Let V be a nontrivial simple tame $\mathbb{F}_2[C]$ -module. Every class in $H_{A,\rho}^1(V)$ has a unique representative supported on x_0 :*

$$(a, b, c, d) \sim (0, 0, c, 0). \quad (5.24)$$

More precisely, the two possible inertia cases are as follows.

1. If $\mathsf{T} = 1$, then $P = 1$, $1 + \mathsf{S}^{-1}$ is invertible, and

$$Z_{A,\rho}^1(V) = \{(a, 0, c, 0) : a, c \in V\}, \quad B_{A,\rho}^1(V) = \{((\mathsf{S} - 1)v, 0, 0, 0) : v \in V\}.$$

2. If $V^\mathsf{T} = 0$, then $P = 0$ and $\mathsf{T} - 1 = 1 + \mathsf{T}$ is invertible. The wild row first forces $d = 0$. Subtracting the coboundary of $v = (\mathsf{T} - 1)^{-1}b$ kills b , and the tame row then forces $a = 0$.

Under the representatives (5.24), the degree-one pairing with the corresponding representative $\lambda \in V^\vee$ is

$$(c, \lambda) \mapsto \begin{cases} \lambda(c), & \mathsf{T} = 1, \\ \lambda((1 + \mathsf{U} + \mathsf{U}^{-1})c), & V^\mathsf{T} = 0. \end{cases} \quad (5.25)$$

Both operators in (5.25) are invertible.

Proof. We reduce separately in the cases of trivial inertia and vanishing inertia fixed space, and then evaluate the pairing on the resulting unique normal forms.

Because tame inertia is normal, V^T is a C -submodule; simplicity gives either $\mathsf{T} = 1$ and $P = 1$, or $V^\mathsf{T} = 0$ and $P = 0$.

If $\mathsf{T} = 1$, the two rows of (5.4) reduce to $b = 0$ and $(1 + \mathsf{S}^{-1})d = 0$. A nonzero kernel of $1 + \mathsf{S}^{-1}$ would be the S -fixed subspace and hence a nonzero submodule, so $1 + \mathsf{S}^{-1}$ is invertible and $d = 0$. The displayed descriptions of Z^1 and B^1 follow. The same argument shows that $\mathsf{S} - 1$ is invertible, so each class has a unique representative $(0, 0, c, 0)$.

If $V^\top = 0$, then $1 + T$ is invertible. The wild row is $S^{-1}d = 0$, hence $d = 0$. For a cocycle $(a, b, c, 0)$, subtract the coboundary of $v = (T - 1)^{-1}b$. The new representative has $b = 0$, and its tame row is $S^{-1}(1 + T)a = 0$, so $a = 0$. Uniqueness follows because a coboundary with zero T -coordinate has $(T - 1)v = 0$, hence $v = 0$.

The pairing formula is [Lemma 5.14](#). In the ramified case U has 2-power order, so its minimal polynomial divides a power of $X + 1$. Since $X^2 + X + 1$ is coprime to $X + 1$,

$$1 + U + U^{-1} = U^{-1}(U^2 + U + 1)$$

is invertible. The unramified operator is the identity. $\square$

Lemma 5.14 (Mixed Hessian calculation). *Let V be a nontrivial simple tame module, put $U = S_2^\omega$, and represent a class in $H_{A,\rho}^1(V)$ by the cocycle supported on x_0 with value c . Represent the dual class similarly by $\lambda \in V^\vee$. The mixed central contributions are*

word contribution	$T = 1$	$V^\top = 0$
r_t	0	0
h_0	$\lambda(c)$	$\lambda(c)$
$[d_0, z_0]$	0	$\lambda(Uc) + \lambda(U^{-1}c)$
all other factors of r_w	0	0.

Consequently the degree-one pairing is induced by

$$\begin{cases} 1, & T = 1, \\ 1 + U + U^{-1}, & V^\top = 0. \end{cases} \quad (5.26)$$

Proof. The representative exists by the row reduction of [\(5.4\)](#). By [Lemma 5.3](#), the word h_0 contributes exactly the usual Heisenberg square diagonal $\lambda(c)$. In the ramified case the first derivatives on the x_0 -coordinate are

$$Dd_0(c) = c, \quad Dz_0(c) = U^{-1}c,$$

whereas on the dual coordinate they are

$$Dd_0(\lambda) = \lambda, \quad Dz_0(\lambda) = U^\vee \lambda.$$

Applying [\(5.7\)](#)–[\(5.8\)](#) to the commutator gives

$$\lambda(U^{-1}c) + (U^\vee \lambda)(c) = \lambda((U^{-1} + U)c).$$

The remaining factors have either zero first derivative on the normalized representatives or no mixed central term. Adding the h_0 contribution proves the table. $\square$

Assembled, these computations say that the candidate's lifting complex satisfies a two-dimensional duality.

Proposition 5.15 (Candidate deformation duality). *For every elementary $\mathbb{F}_2[C]$ -module A , the chain map [\(5.17\)](#) is a quasi-isomorphism. Hence*

$$H_{A,\rho}^2(A)^\vee \cong (A^\vee)^C, \quad \dim Z_{A,\rho}^1(A) = 2 \dim A + \dim (A^\vee)^C. \quad (5.27)$$

The induced pairings

$$H_{A,\rho}^i(A) \times H_{A,\rho}^{2-i}(A^\vee) \rightarrow \mathbb{F}_2$$

are perfect and natural in A .

Proof. We treat the trivial simple module, then the nontrivial simple modules, and finally pass to arbitrary elementary modules by exact cone dévissage.

For $A = \mathbb{F}_2$, all lower actions are trivial, so $d^0 = 0$ and the actual Jacobian is

$$d^1(a, b, c, d) = (b, b).$$

Thus $\dim H^0 = 1$, $\dim H^1 = 3$, and $\dim H^2 = 1$. On the basis of H^1 represented by the coordinates (σ, x_0, x_1) , the traced Fox–Heisenberg form is

$$((a, c, d), (a', c', d')) \mapsto ad' + da' + cc',$$

with matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is nonsingular, and the endpoint trace gives a nonzero H^0 – H^2 pairing. This is exactly the scalar cup–Bockstein table (3.18).

For a nontrivial simple module, Lemma 5.12 makes the action tame, and Lemma 5.13 gives a unique degree-one representative supported on x_0 . The Jacobian is surjective, so $H^0 = H^2 = 0$ and $\dim H^1 = \dim A$. On the unique normal forms the degree-one pairing is exactly (5.25), and is nonsingular by the final assertion of Lemma 5.13. Hence the cone $K(A)$ is acyclic for every simple module, including the trivial simple module treated above.

For a general elementary module, choose a composition series. The degreewise short exact cone sequence (5.21) and its long exact cohomology sequence give acyclicity at each extension step. Thus $D_{\rho, A}$ is a quasi-isomorphism for every elementary, possibly nonsplit and nonsemisimple, module A . The numerical formula follows from duality and the Euler characteristic of (5.1). $\square$

The Galois side satisfies the same duality, with the local cup product and invariant map supplying the pairing.

Proposition 5.16 (Local lifting duality). *For every elementary $\mathbb{F}_2[C]$ -module A ,*

$$H_{F, \rho}^2(A)^\vee \cong (A^\vee)^C, \quad \dim Z_{F, \rho}^1(A) = 2 \dim A + \dim(A^\vee)^C. \quad (5.28)$$

The local cup product and invariant map give perfect pairings

$$H_{F, \rho}^i(A) \times H_{F, \rho}^{2-i}(A^\vee) \rightarrow \mathbb{F}_2.$$

Proof. This is local Tate duality, with the mod-2 cyclotomic twist trivial, together with the local Euler–Poincaré formula for $\mathbb{Q}_2$; see [1, Chapter VII, Sections 2–3]. Since the action factors through C , the invariant space is $(A^\vee)^C$. Finally

$$\dim Z^1 = \dim H^1 + \dim B^1 = \dim H^1 + \dim A - \dim A^C,$$

and the Euler characteristic gives the stated expression. $\square$

Corollary 5.17 (Common lift numerics and adjoint boundaries). *For either source, use the corresponding notation $H_{\Gamma, \rho}^i$ and $Z_{\Gamma, \rho}^1$. Then the obstruction-space dimensions, unobstructed lift multiplicities, and perfect pairings agree numerically.*

For every short exact sequence

$$0 \rightarrow T \rightarrow M \rightarrow V \rightarrow 0$$

and its dual, the connecting maps are adjoint: if

$$\partial_{\Gamma, \rho} : H_{\Gamma, \rho}^1(V) \rightarrow H_{\Gamma, \rho}^2(T)$$

and $\chi \in (T^\vee)^C$ has dual boundary

$$\gamma_\chi = \partial_C^\vee(\chi) \in H^1(C, V^\vee),$$

then

$$\langle \partial_{\Gamma, \rho} c, \chi \rangle = \langle c, \rho^* \gamma_\chi \rangle_\Gamma. \quad (5.29)$$

Proof. For the candidate, (5.29) is the naturality of the chain map (5.17) applied to the coefficient sequence. For the local source it is the compatibility of connecting homomorphisms with cup product and the local invariant map in the Tate duality pairing used in Proposition 5.16. The numerical assertions are Proposition 5.15, Proposition 5.16. $\square$

6 Quadratic determinant obstructions

The Evens norm and the Stiefel–Whitney comparison used throughout this section follow [9], [10], and [11]; for the computation of Stiefel–Whitney classes see also [6].

The linear interface of Section 5 does not determine the scalar central pushouts attached to self-dual simple modules, because their lifting obstruction is quadratic. We construct a normalized equivariant base class, compute the resulting form on the candidate and local sides, compare their Gauss sums, and then restore the affine terms required in the finite-target recursion.

Determinant obstruction. Let

$$1 \longrightarrow R \longrightarrow \widetilde{M} \longrightarrow M \longrightarrow 1$$

be a C -equivariant central extension with $\widetilde{M}$ of class at most two and with R and M elementary abelian, and let $\lambda \in (R^\vee)^C$. Pushing out along $\lambda : R \rightarrow \mathbb{F}_2$ gives a central double cover of M . Its square map

$$q_\lambda(m) = \lambda(\widetilde{m}^2) \in \mathbb{F}_2$$

is quadratic, with polar form given by the commutator pairing. The scalar degree-two obstruction obtained by pulling back this cover is called the *determinant obstruction*. In the application, its restriction to the nontrivial simple head V is a nonzero nonsingular quadratic form; this is proved in Proposition 7.4.

The notation used for the quadratic and affine terms is as follows.

Symbol	Meaning
q	a C -invariant nonsingular quadratic form on the simple self-dual module V
κ_q^0	the zero-section-normalized equivariant central 2-cocycle on $V \rtimes C$ whose fibre square map is q
γ	the filtration-one class in $Z^1(C, V^\vee)$; its contribution on $V \rtimes C$ is the cocycle Γ_γ
δ	the scalar class in $Z^2(C, \mathbb{F}_2)$ inflated from the lower quotient C
e	the class in $H^2(C, T)$ of the affine T -extension governing which V -cocycles lift through M
$\Delta_{\chi, \kappa}$	the scalar class $\chi_* e + \delta_\kappa + \Theta_q^0(a_{\chi, \kappa})$ obtained after completing the square

For the candidate source we write

$$\iota_A : H_{A,\rho}^2(\mathbb{F}_2) \longrightarrow \mathbb{F}_2$$

for the functional induced by the degree-two trace

$$D^2(u_t, u_w) = u_t + u_w$$

of [Proposition 5.10](#). For the local source we write

$$\iota_F = \text{inv}_{\mathbb{Q}_2} : H^2(G_{\mathbb{Q}_2}, \mathbb{F}_2) \longrightarrow \mathbb{F}_2.$$

All quadratic obstruction functions below use the source-specific functional ι_Γ .

6.1 The normalized equivariant base class

A C -invariant quadratic form on V need not admit a C -invariant factor set for a chosen section $V \rightarrow E$. The correction terms in the lifted C -action are therefore included in the equivariant central extension class.

Lemma 6.1 (Equivariant extraspecial connecting cocycle). *Let V be an elementary $\mathbb{F}_2[C]$ -module, let $q : V \rightarrow \mathbb{F}_2$ be a C -invariant quadratic form with polar form b_q , and let*

$$f : V \times V \rightarrow \mathbb{F}_2, \quad f(v, v) = q(v), \quad f(v, w) + f(w, v) = b_q(v, w),$$

be a normalized factor set for the central extension

$$E_f = V \times \mathbb{F}_2, \quad (v, z)(w, t) = (v + w, z + t + f(v, w)).$$

When an equivariant lift is chosen, its action in the normalized section is written by functions $m_c : V \rightarrow \mathbb{F}_2$ satisfying

$$m_c(v + w) + m_c(v) + m_c(w) = f(cv, cw) + f(v, w) \tag{6.1}$$

and

$$m_{cd}(v) = m_c(dv) + m_d(v), \quad m_1 = 0. \tag{6.2}$$

Equivalently,

$$c \cdot (v, z) = (cv, z + m_c(v))$$

defines an action by automorphisms of E_f . The associated normalized base central cocycle on $V \rtimes C$ is

$$\kappa_q^0((v, c), (w, d)) = f(v, cw) + m_c(w). \tag{6.3}$$

Consequently, for a source Γ , a lower map $\rho : \Gamma \rightarrow C$, and a normalized V -valued 1-cocycle b , the base connecting cocycle is

$$(b, \rho)^* \kappa_q^0(g, h) = f(b(g), \rho(g)b(h)) + m_{\rho(g)}(b(h)). \tag{6.4}$$

If κ is another normalized central cocycle on $V \rtimes C$ whose restriction to V is the same quadratic form q , then, after replacing κ by a cohomologous normalized cocycle and changing the normalized section, the following is an equality of normalized 2-cochains:

$$\kappa = \kappa_q^0 + \Gamma_\gamma + \inf_C^{V \rtimes C} \delta, \tag{6.5}$$

where

$$\Gamma_\gamma((v, c), (w, d)) = \gamma(c)(cw), \quad \gamma \in Z^1(C, V^\vee), \quad \delta \in Z^2(C, \mathbb{F}_2). \quad (6.6)$$

After graph pullback and application of the source functional ι_Γ , the corresponding scalar quadratic obstruction is

$$Q_{\kappa, \Gamma, \rho}(b) = Q_{\Gamma, \rho}^0(b) + \langle b, \rho^* \gamma \rangle_\Gamma + \iota_\Gamma(\rho^* \delta), \quad (6.7)$$

where

$$Q_{\Gamma, \rho}^0(b) = \iota_\Gamma((b, \rho)^* \kappa_q^0).$$

Its polarization is the source pairing transported by b_q .

Proof. We verify the equivariant central extension, identify the filtration-one and inflated difference terms, and then compute the polarization.

Equation (6.1) is exactly the condition that $(v, z) \mapsto (cv, z + m_c(v))$ preserve the multiplication in E_f ; equation (6.2) says that these automorphisms form a genuine C -action. Multiplication in the semidirect product attached to this lifted action is

$$(v, c, z)(w, d, t) = (v + cw, cd, z + t + f(v, cw) + m_c(w)), \quad (6.8)$$

which gives (6.3). Pulling back this cocycle by the graph of a 1-cocycle b gives (6.4). This is the term which would be missing if one incorrectly assumed that the section $v \mapsto (v, 0)$ were C -equivariant.

For the difference statement, the quotient $\kappa - \kappa_q^0$ restricts trivially to $V \times V$. In the normalized Lyndon–Hochschild–Serre filtration for the split extension $1 \rightarrow V \rightarrow V \rtimes C \rightarrow C \rightarrow 1$, such a class has only a filtration-one part in $Z^1(C, V^\vee)$ and an inflated scalar part in $Z^2(C, \mathbb{F}_2)$. The normalized representative is exactly (6.6); the cocycle condition for it is the condition that γ be a 1-cocycle. Pullback along (b, ρ) gives (6.7). Polarization of the base term is obtained by adding the formulas for $b + b'$, b , and b' ; the m_c -contribution cancels because each m_c has polar form $f(c-, c-) + f(-, -)$, and the remaining term is $b_q(b \smile b')$. The difference terms in (6.7) are linear or inflated scalar terms and therefore do not change the polar form. $\square$

Lemma 6.2 (Explicit involution-orbit cocycle). *Let H be a finite group, let $g \in H$ have order 2, and let $M = \mathbb{F}_2[H]$ with left-regular action*

$$(cx)_h = x_{c^{-1}h}.$$

Choose a set $\mathcal{R} \subset H$ of representatives for the right cosets $u\langle g \rangle$. For $c \in H$ and $u \in \mathcal{R}$, write uniquely

$$c^{-1}u = \pi_c(u)g^{\epsilon_c(u)}, \quad \pi_c(u) \in \mathcal{R}, \quad \epsilon_c(u) \in \mathbb{F}_2. \quad (6.9)$$

Define

$$q_g(x) = \sum_{u \in \mathcal{R}} x_u x_{ug}, \quad (6.10)$$

$$f_g(x, y) = \sum_{u \in \mathcal{R}} x_u y_{ug}, \quad (6.11)$$

$$m_c^g(x) = \sum_{u \in \mathcal{R}} \epsilon_c(u) x_{\pi_c(u)} x_{\pi_c(u)g}. \quad (6.12)$$

Then

$$m_c^g(x+y) + m_c^g(x) + m_c^g(y) = f_g(cx, cy) + f_g(x, y), \quad (6.13)$$

$$m_{cd}^g(x) = m_c^g(dx) + m_d^g(x), \quad m_1^g = 0. \quad (6.14)$$

Consequently

$$\kappa_g^{\mathcal{R}}((x, c), (y, d)) = f_g(x, cy) + m_c^g(y) \quad (6.15)$$

is a normalized 2-cocycle on $M \rtimes H$. It vanishes on the zero section $H \hookrightarrow M \rtimes H$, and its restriction to M has square map q_g .

Proof. We first compute the transition functions for the right-coset action, then verify the quadratic correction, and finally apply the general equivariant extension formula.

From (6.9) one obtains

$$\pi_{cd} = \pi_d \circ \pi_c, \quad \epsilon_{cd}(u) = \epsilon_c(u) + \epsilon_d(\pi_c(u)). \quad (6.16)$$

Moreover,

$$f_g(cx, cy) = \sum_{u \in \mathcal{R}} x_{\pi_c(u)g^{\epsilon_c(u)}} y_{\pi_c(u)g^{\epsilon_c(u)+1}}.$$

For $\epsilon_c(u) = 0$ this is the summand of $f_g(x, y)$ indexed by $\pi_c(u)$; for $\epsilon_c(u) = 1$ its orientation is reversed. Therefore

$$f_g(cx, cy) + f_g(x, y) = \sum_{u \in \mathcal{R}} \epsilon_c(u) (x_{\pi_c(u)} y_{\pi_c(u)g} + x_{\pi_c(u)g} y_{\pi_c(u)}),$$

which is exactly the polarization of (6.12). This proves (6.13).

The product $x_v x_{vg}$ is unchanged when v is replaced by vg . Hence

$$m_c^g(dx) = \sum_{u \in \mathcal{R}} \epsilon_c(u) x_{\pi_d(\pi_c(u))} x_{\pi_d(\pi_c(u))g}.$$

Adding $m_d^g(x)$ and using (6.16) gives (6.14). The cocycle assertion now follows directly from Lemma 6.1. Finally, $f_g(x, x) = q_g(x)$ and $m_1^g = 0$, proving the restriction and normalization claims. $\square$

Lemma 6.3 (Base equivariant determinant class). *Let V be a nontrivial simple self-dual tame $\mathbb{F}_2[C]$ -module, let $H = H_V$ be its faithful tame image, and let $q : V \rightarrow \mathbb{F}_2$ be a nonzero H -invariant nonsingular quadratic form. There is a normalized base class*

$$\kappa_q^0 \in Z^2(V \rtimes H, \mathbb{F}_2)$$

whose restriction to V has square map q .

Choose an H -split pair

$$V \xrightarrow{i} W = \mathbb{F}_2[H]^N \xrightarrow{p} V, \quad pi = 1,$$

and put $q_W = q \circ p$. In the left-regular coordinates $X_{j,h}$ of W , the unique algebraic normal form of the H -invariant quadratic map q_W is a sum of orbit polynomials of the following three types:

$$S_j = \sum_{h \in H} X_{j,h}^2,$$

$$C_{j,k,g} = \sum_{h \in H} X_{j,h} X_{k,hg} \quad \text{when the corresponding unordered pair has trivial stabilizer,}$$

$$E_{j,g} = \sum_{u \in \mathcal{R}_g} X_{j,u} X_{j,ug}, \quad g^2 = 1, \quad g \neq 1,$$

where $\mathcal{R}_g$ is any set of representatives for the right cosets of $\langle g \rangle$ in H . For a square orbit use the H -invariant factor set

$$f_{S_j}(x, y) = \sum_{h \in H} x_{j,h} y_{j,h}, \quad m_c = 0. \quad (6.17)$$

For a free orbit use

$$f_{C_{j,k,g}}(x, y) = \sum_{h \in H} x_{j,h} y_{k,hg}, \quad m_c = 0. \quad (6.18)$$

For an involution-stabilized orbit use the cocycle (6.15) of Lemma 6.2 on the j th regular summand, with $\mathcal{R} = \mathcal{R}_g$. Let κ_{qW}^0 be the sum of the cocycles corresponding to the orbit polynomials occurring in qW , and put

$$\kappa_q^0 = (i \rtimes 1)^* \kappa_{qW}^0. \quad (6.19)$$

Then κ_q^0 vanishes on the zero section and $\text{res}_V(\kappa_q^0)$ has square map q . If a lower target has quotient C acting through H , we inflate this class along $C \rightarrow H$ and keep the same notation.

Proof. We construct the class on a split regular module by the three possible orbit types of quadratic monomials and then pull it back to V .

If V is ramified, V is projective over $\mathbb{F}_2[H]$ by Lemma 6.11; if V is unramified, the 2-primary subgroup of the cyclic unramified image acts trivially on every simple characteristic-2 module, so the faithful image H has odd order and Maschke's theorem applies. Thus the split pair exists.

The algebraic normal form of a quadratic map over $\mathbb{F}_2$ is determined by its values on the coordinate vectors and by its polar form. H -invariance makes its coefficients constant on the orbits of unordered degree-two coordinate monomials. Since the action on each regular coordinate set is free, the stabilizer of an unordered pair is either trivial or has order two and exchanges the two entries. The latter possibility is exactly $j = k$ with a nontrivial involution g , and gives $E_{j,g}$.

The factor sets (6.17) and (6.18) are H -invariant, vanish on the zero section, and have square maps S_j and $C_{j,k,g}$, respectively. By Lemma 6.2, the explicit cocycle for $E_{j,g}$ has square map $E_{j,g}$ and also vanishes on the zero section. Their sum is therefore an actual normalized equivariant central cocycle on $W \rtimes H$ with fibre square map q_W . Pullback along $i \rtimes 1$ gives fibre square map $q_W(iv) = q(v)$ and preserves zero-section normalization. $\square$

Lemma 6.4 (Uniqueness and isometry invariance of the normalized base class). *Let V , $H = H_V$, and q be as in Lemma 6.3. Then*

$$H^1(H, V^\vee) = 0. \quad (6.20)$$

Any two classes in $H^2(V \rtimes H, \mathbb{F}_2)$ which restrict to the class with square map q on V and restrict trivially along the zero section $H \hookrightarrow V \rtimes H$ are equal. Consequently, changing the normalized factor set, the correction functions m_c , the half-orbit transversals, or the H -split regular embedding and retraction changes κ_q^0 only by a normalized coboundary. In particular the base quadratic obstruction $Q_{\Gamma,\rho}^0$ is well-defined.

Moreover, every H -equivariant isometry u of (V, q) fixes the normalized base class:

$$(u \rtimes 1)^* [\kappa_q^0] = [\kappa_q^0]. \quad (6.21)$$

Proof. We first prove the required cohomology vanishing, then use the low-degree Lyndon–Hochschild–Serre sequence for uniqueness, and finally deduce choice-independence and isometry invariance.

If V is unramified, its faithful image H has odd order: the 2-primary subgroup of the cyclic unramified image acts trivially on every simple characteristic-2 module. Hence positive-degree $\mathbb{F}_2[H]$ -cohomology vanishes by averaging.

Assume that V is ramified, and let $I \triangleleft H$ be the odd tame-inertia subgroup. The fixed space $(V^\vee)^I$ is an H -submodule. It is zero, because V is simple and inertia acts nontrivially. Since $|I|$ is odd, $H^j(I, V^\vee) = 0$ for $j > 0$. The Hochschild–Serre spectral sequence for $1 \rightarrow I \rightarrow H \rightarrow H/I \rightarrow 1$ therefore gives [5, Chapter VII]

$$H^n(H, V^\vee) = 0 \quad (n \geq 0).$$

In particular (6.20) holds in both cases.

Put $G = V \rtimes H$. For an elementary abelian 2-group V , the square-map correspondence

$$H^2(V, \mathbb{F}_2) \xrightarrow{\sim} \{\text{quadratic maps } V \rightarrow \mathbb{F}_2\}$$

is an isomorphism: the polarization recovers the commutator form, and the square and commutator data determine the central extension class. Thus two classes with square map q have the same fibre restriction.

The low-degree Lyndon–Hochschild–Serre sequence for $1 \rightarrow V \rightarrow G \rightarrow H \rightarrow 1$ contains the exact segment [5, Chapter VII]

$$H^2(H, \mathbb{F}_2) \xrightarrow{\inf} \ker(H^2(G, \mathbb{F}_2) \longrightarrow H^2(V, \mathbb{F}_2)) \xrightarrow{\text{edge}} H^1(H, V^\vee). \quad (6.22)$$

Let ξ be the difference of two classes with the stated normalizations. Its fibre restriction is zero. By (6.20) and exactness of (6.22), $\xi = \inf(\delta)$ for some $\delta \in H^2(H, \mathbb{F}_2)$. Pullback along the zero section is a left inverse to inflation, while both classes vanish there. Hence $\delta = 0$ and $\xi = 0$.

At the filtration-one level one can see the same cancellation directly. If $\gamma = dh$ for $h \in V^\vee$, then the normalized 1-cochain

$$\ell_h(v, c) = h(v) \quad (6.23)$$

on $V \rtimes H$ satisfies $d\ell_h = \Gamma_{dh}$, with Γ as in (6.6).

If normalized cocycle representatives are used, their difference is da for a normalized 1-cochain a . Since the difference vanishes on $H \times H$, $a|_H$ is a 1-cocycle. Subtracting its inflation makes a vanish on the zero section without changing da . Thus all choices in Lemma 6.3 give isomorphic normalized central extensions, and transport of lifts through that isomorphism leaves both the candidate relator obstruction and the local pullback class unchanged. This proves the choice-independence of $Q_{\Gamma, \rho}^0$.

Finally, $(u \rtimes 1)^* \kappa_q^0$ has the same fibre square map and the same zero-section restriction as κ_q^0 , so uniqueness proves (6.21).

The uniqueness assertion is deliberately made over the faithful image H_V . After inflation to a larger quotient C acting through H_V , the group $H^1(C, V^\vee)$ need not vanish; its contribution is precisely the filtration-one term Γ_γ in (6.5), and it remains part of the target-side affine data. $\square$

6.2 Candidate-side calculation

Let V be a nontrivial self-dual simple tame module, let $q : V \rightarrow \mathbb{F}_2$ be invariant and nonsingular, let b_q be its polar form, and let κ_q^0 be the base equivariant determinant class of Lemma 6.3. The candidate determinant calculation below is made in the lifted semidirect product (6.8); thus all m_c -terms are included.

Proposition 6.5 (Complete base second-order word expansion). *With $S, T, P, U = S_2^\omega$ as above, evaluation of the actual relator (1.6) for the base class κ_q^0 gives*

$$Q_A^0(c) = q(c) + b_q((P+1)c, U^{-1}c). \quad (6.24)$$

Equivalently,

$$Q_A^0(c) = \begin{cases} q(c), & T = 1, \\ q(c) + b_q(c, U^{-1}c), & V^T = 0. \end{cases} \quad (6.25)$$

Its ramified polar form is

$$b_A(c, c') = b_q(c, (1 + U + U^{-1})c'). \quad (6.26)$$

For a general determinant class $\kappa = \kappa_q^0 + \Gamma_{\gamma_\kappa} + \inf \delta_\kappa$, one has

$$Q_{A, \kappa, \rho}(c) = Q_A^0(c) + \langle c, \rho^* \gamma_\kappa \rangle_A + \iota_A(\rho^* \delta_\kappa). \quad (6.27)$$

Proof. We evaluate the square contribution of h_0 , the commutator contribution of d_0 and z_0 , and then polarize the resulting quadratic expression.

Normalize a cohomology representative so that only x_0 varies, by c . In terms of (6.8), the complete base contribution ledger is

factor	V-coordinate	central coordinate
h_0	0	$q(c)$
u_1^{-1}	0	0
x_1^σ	0	0
$d_0 = (x_0 \tau)_2^\omega x_0^{-1}$	$(P+1)c$	*
$z_0 = x_0^{\sigma_2^\omega}$	$U^{-1}c$	*
$[d_0, z_0]$	0	$b_q((P+1)c, U^{-1}c)$

The starred entries contain ordinary f -terms and the m -terms of the lifted C -action. They occur only as central coordinates of d_0 and z_0 , and central coordinates do not affect the commutator. Since the lower values of d_0 and z_0 at the tame map are both the identity, the commutator contribution is exactly the polar form of their V -coordinates, $b_q((P+1)c, U^{-1}c)$.

The first line is the extraspecial case of Lemma 5.3, but we recall why it remains valid with the m -terms present. The class-two identity Lemma 5.2 uses only that the lift of the relevant tame operator is an automorphism of the central extension and fixes the centre; it does not assume that the chosen section is equivariant. If $P = 0$, then d_0 and x_0 have the same image in V , so the identity gives the square of x_0 , namely $q(c)$. If $P = 1$, then $T = 1$ and the lift of U is a central automorphism of order at most 2, so $g_0 = \sigma_2^2$ acts trivially on the class-two coefficient group and the same identity applies.

The factors u_1^{-1} and x_1^σ have zero V -coordinate on the normalized representative, and hence no quadratic central contribution. Multiplying the table gives (6.24). Polarizing and using the U -invariance of b_q gives (6.26); the operator is invertible by the same polynomial argument used in Proposition 5.15. Finally, (6.27) is exactly (6.7) on the candidate source. $\square$

6.2.1 The Arf invariant of the candidate form

Lemma 6.6 (Wall doubling). *Let (V, q) be nonsingular over $\mathbb{F}_2$, with polar form b_q , and let $U \in O(V, q)$ have 2-power order. Put*

$$q_U(x) = q(x) + b_q(x, U^{-1}x).$$

Then q_U is nonsingular and

$$\text{Arf}(q_U) = \text{Arf}(q) + \text{rank}(1 + U) \pmod{2}. \quad (6.28)$$

Proof. Let $N = 1 + U$, $R = NV$, and $K = \ker N$. Since $q(Ux) = q(x)$,

$$b_q(x, U^{-1}x) = b_q(x, Ux) = q(Nx),$$

so $q_U(x) = q(x) + q(Nx)$. Define the nondegenerate Wall form on R by

$$\omega(Nx, u) = b_q(x, u).$$

In $(V, q_U) \perp (V, q)$, the diagonal $\{(k, k) : k \in K\}$ is totally singular. Its orthogonal quotient is $R \oplus R$ with quadratic form

$$\Phi_\omega(v, u) = \omega(v, v) + \omega(u, u) + \omega(v, u).$$

If $v \mapsto \omega(v, v)$ is nonzero, split an anisotropic plane from Φ_ω ; if ω is alternating, split a symplectic plane from R , which contributes a hyperbolic four-space. Induction gives $\text{Arf}(\Phi_\omega) = \dim R \pmod{2}$, proving (6.28). $\square$

Lemma 6.7 (Invariant quadratic forms on a hermitian line). *Let F/F_0 be a quadratic extension of finite fields of characteristic 2, write $x \mapsto x^*$ for its involution, and let $\mathcal{U}_1 = \{u \in F^\times : uu^* = 1\}$. Let W be a one-dimensional F -space. Every nonsingular $\mathcal{U}_1$ -invariant quadratic form $Q : W \rightarrow \mathbb{F}_2$ is uniquely of the form*

$$Q(x) = \text{Tr}_{F_0/\mathbb{F}_2}(axx^*) \quad (a \in F_0^\times).$$

Proof. Choose a generator u_0 of the cyclic group $\mathcal{U}_1$. Since $2^m \equiv -1 \pmod{2^m + 1}$ and no smaller positive power of 2 is 1 modulo $2^m + 1$, the element u_0 has degree $2m$ over $\mathbb{F}_2$; hence $\mathbb{F}_2[u_0] = F$. Invariance gives

$$b_Q(dx, y) = b_Q(x, d^*y) \quad (d \in F).$$

Putting $\ell(y) = b_Q(1, y)$, we obtain $b_Q(x, y) = \ell(x^*y)$. Write $\ell(z) = \text{Tr}_{F/\mathbb{F}_2}(cz)$ for the unique $c \in F$. Symmetry of b_Q forces $c = c^*$, hence $c \in F_0$, and therefore

$$b_Q(x, y) = \text{Tr}_{F_0/\mathbb{F}_2}(c(xy^* + x^*y)).$$

Conversely these forms are alternating, and they are nonsingular exactly when $c \neq 0$. The quadratic form $q_c(x) = \text{Tr}_{F_0/\mathbb{F}_2}(cxx^*)$ has this polarization. Hence $Q - q_c$ is a $\mathcal{U}_1$ -invariant linear form. Writing such a form as $x \mapsto \text{Tr}_{F/\mathbb{F}_2}(bx)$, invariance under u_0 gives $b(u_0 - 1) = 0$, so $b = 0$. Thus $Q = q_c$, uniquely, and nonsingularity gives $c \in F_0^\times$. $\square$

Lemma 6.8 (Ramified Hermitian model and Frobenius fixed space). *Let $I = \langle T \rangle$ be tame inertia, let V be a simple ramified self-dual $\mathbb{F}_2[H_V]$ -module, and write*

$$V|_I \cong W^{\oplus s}, \quad f = \dim_{\mathbb{F}_2} W = 2^a r, \quad r \text{ odd}, \quad a \geq 1.$$

Let q be an H_V -invariant nonsingular quadratic form with polar form b_q . Then

$$\text{Arf}(q) \equiv s \pmod{2}. \quad (6.29)$$

For $U = S^{\omega_2}$ one also has

$$\dim_{\mathbb{F}_2} V^U = rs, \quad \text{rank}(1 + U) = rs(2^a - 1) \equiv s \pmod{2}. \quad (6.30)$$

Consequently the ramified candidate base form of (6.25) has $\text{Arf}(Q_A^0) = 0$.

Proof. We identify the invariant quadratic form with a Hermitian form, compute its Arf invariant one Hermitian line at a time, and then determine the fixed space of the 2-primary Frobenius operator.

Put

$$F_V = \text{End}_I(W) \cong \mathbb{F}_{2^f}.$$

Choose an inertia character ζ over $\overline{\mathbb{F}_2}$; then W is the additive group of F_V and T acts by multiplication by ζ . Since $V|_I$ has only the irreducible isomorphism type W and $V \cong V^\vee$, the inverse character ζ^{-1} lies in the Frobenius orbit of ζ . The character is nontrivial of odd order, so this orbit has even length and

$$\zeta^{-1} = \zeta^{2^{f/2}}.$$

Set

$$F_{V,0} = \mathbb{F}_{2^{f/2}}, \quad x^* = x^{2^{f/2}}.$$

After an I -module identification $V \cong F_V^s$, the nondegenerate I -invariant form b_q is uniquely expressible as

$$b_q(x, y) = \text{Tr}_{F_V/\mathbb{F}_2}(x^{*t} H y), \quad H \in \text{GL}_s(F_V), \quad H^{*t} = H. \quad (6.31)$$

Indeed, invariance under multiplication by ζ , nondegeneracy of the trace pairing, and symmetry identify the adjoint involution with $*$ and force the matrix to be Hermitian.

Define

$$q_H(x) = \text{Tr}_{F_{V,0}/\mathbb{F}_2}(x^{*t} H x). \quad (6.32)$$

The scalar $x^{*t} H x$ lies in $F_{V,0}$, and

$$\begin{aligned} q_H(x + y) + q_H(x) + q_H(y) &= \text{Tr}_{F_{V,0}/\mathbb{F}_2}(x^{*t} H y + y^{*t} H x) \\ &= \text{Tr}_{F_V/\mathbb{F}_2}(x^{*t} H y) = b_q(x, y). \end{aligned}$$

Thus $q - q_H$ is linear. It is I -invariant, while $(V^\vee)^I = 0$ because the inertia action is nontrivial and V is simple. Hence $q = q_H$.

Hermitian diagonalization over the finite field extension $F_V/F_{V,0}$ gives

$$H \sim \text{diag}(a_1, \dots, a_s), \quad a_i \in F_{V,0}^\times.$$

Therefore q is an orthogonal sum of the s Hermitian-line forms

$$q_{a_i}(x) = \text{Tr}_{F_{V,0}/\mathbb{F}_2}(a_i x x^*).$$

Write $m = f/2$. The norm $F_V^\times \rightarrow F_{V,0}^\times$ is surjective with fibres of size $2^m + 1$, and the trace-zero subspace of $F_{V,0}$ contains $2^{m-1} - 1$ nonzero elements. Hence

$$\#q_{a_i}^{-1}(0) = 1 + (2^m + 1)(2^{m-1} - 1) = 2^{2m-1} - 2^{m-1}.$$

Each Hermitian line has Arf invariant 1. Additivity of the Arf invariant proves (6.29).

It remains to compute the fixed space of U . Over $\mathbb{F}_2$, the f inertia weights form one Frobenius cycle, each weight space having multiplicity s . Let e_ω be the residue modulo $f = 2^a r$ of an integer representing ω_2 . By the defining congruences of ω_2 ,

$$e_\omega \equiv 1 \pmod{2^a}, \quad e_\omega \equiv 0 \pmod{r}.$$

Thus translation by e_ω on $\mathbb{Z}/f\mathbb{Z}$ has r cycles, each of length 2^a . On one cycle write the successive maps as

$$E_0 \xrightarrow{A_0} E_1 \xrightarrow{A_1} \cdots \xrightarrow{A_{2^a-1}} E_0, \quad \dim E_i = s.$$

The product $A_{2^a-1} \cdots A_0$ is the action of U^{2^a} on E_0 . The shift by $2^a e_\omega$ is trivial modulo f , so U^{2^a} centralizes inertia; it also commutes with S , being a power of S . Hence it is a central 2-element of the faithful tame image. A cyclic 2-group acting in characteristic 2 has nonzero fixed vectors, and centrality makes its fixed space an H_V -submodule. Simplicity therefore forces $U^{2^a} = 1$ on V . Consequently a U -fixed vector on a cycle is uniquely determined by its component in E_0 , and that cycle contributes dimension s . There are r cycles, proving (6.30).

Finally, Lemma 6.6 gives

$$\text{Arf}(Q_A^0) = \text{Arf}(q) + \text{rank}(1 + U) = s + s = 0 \quad \text{in } \mathbb{F}_2.$$

□

These determinant computations combine into the candidate-side zero count.

Proposition 6.9 (Candidate base determinant zero count). *If $d = \dim V$, then*

$$\#(Q_A^0)^{-1}(0) = \begin{cases} 2^{d-1} - 2^{d/2-1}, & V \text{ unramified,} \\ 2^{d-1} + 2^{d/2-1}, & V \text{ ramified.} \end{cases} \quad (6.33)$$

Proof. We determine the unramified and ramified signs separately, using the Hermitian-line count in the first case and Lemma 6.8 in the second.

If V is unramified, $U = 1$ and $Q_A^0 = q$. Put $F_V = \text{End}_{H_V}(V) \cong \mathbb{F}_{2^{2m}}$ and let $F_{V,0} \cong \mathbb{F}_{2^m}$ be the fixed field of the adjoint involution $d \mapsto d^*$ defined by the polar form b_q . The simple cyclic module V is a one-dimensional F_V -space. The norm-one group

$$\mathcal{U} = \{u \in F_V^\times : uu^* = 1\}$$

preserves b_q . For $u \in \mathcal{U}$, the function $x \mapsto q(ux) + q(x)$ is therefore linear. It is H_V -invariant because u commutes with H_V and q is H_V -invariant. Since V is nontrivial simple, $(V^\vee)^{H_V} = 0$, so the function is zero. Thus $\mathcal{U} \leq O(V, q)$.

By Lemma 6.7,

$$q(x) = \text{Tr}_{F_{V,0}/\mathbb{F}_2}(axx^*), \quad a \in F_{V,0}^\times.$$

The norm fibres have size $2^m + 1$, so the number of zeros is $2^{2m-1} - 2^{m-1}$ and the Arf invariant is 1.

Assume V is ramified. The complete Hermitian and fixed-space calculation is [Lemma 6.8](#); in particular $\text{Arf}(Q_A^0) = 0$. For a nonsingular quadratic form of dimension d , the identity

$$\#Q^{-1}(0) = 2^{d-1} + (-1)^{\text{Arf}(Q)} 2^{d/2-1}$$

now gives [\(6.33\)](#). □

6.3 Local calculation

This subsection uses the unit-filtration structure of dyadic local fields [\[7, Chapter IV §2\]](#), the norm criteria for quadratic extensions [\[7, Chapter V §2 and Chapter XIV §2\]](#), and local Tate duality [\[1, Theorem \(7.2.6\)\]](#).

Let $q : V \rightarrow \mathbb{F}_2$ be the invariant nonsingular quadratic form under consideration, and let κ_q^0 be the base equivariant determinant class of [Lemma 6.3](#). For a local cocycle $b : G_{\mathbb{Q}_2} \rightarrow V$ with lower map ρ , put

$$Q_{\text{loc}}^0([b]) = \text{inv}_{\mathbb{Q}_2}[(b, \rho)^* \kappa_q^0]. \quad (6.34)$$

This is the base quadratic connecting map. For a general determinant pushout class $\kappa = \kappa_q^0 + \Gamma_{\gamma_\kappa} + \text{inf } \delta_\kappa$, the actual local obstruction is obtained from [\(6.34\)](#) by the difference formula [\(6.7\)](#); the affine terms are not folded into the base Gauss sum.

Let H_V be the faithful tame image on V and let $K/\mathbb{Q}_2$ be its tame splitting field. Put $M_K = K^\times / K^{\times 2}$ and let U_i be the image of $1 + \mathfrak{p}_K^i$. If $e = v_K(2)$, then

$$\text{gr}_i M_K \cong \begin{cases} k_K, & 1 \leq i < 2e \text{ odd}, \\ 0, & 1 \leq i < 2e \text{ even}, \end{cases} \quad (6.35)$$

and

$$U_i^\perp = U_{2e-i+1}, \quad -1 \in U_e. \quad (6.36)$$

These are the standard dyadic square-class filtration and Hilbert-symbol orthogonality relations; see [\[7, Chapter XIV, Sections 2–3\]](#).

Lemma 6.10 (The middle unit layer is unramified). *Assume $K/\mathbb{Q}_2$ is tame and put $e = v_K(2)$. Then e is odd, and tame inertia acts trivially on the middle graded piece*

$$U_e/U_{e+1}.$$

Consequently a ramified simple tame module does not occur in that graded piece.

Proof. The ramification index of a tame extension of $\mathbb{Q}_2$ is odd and equals the order of tame inertia. On the odd-depth piece $\text{gr}_j M_K$, an inertia element represented by ζ acts through the character $\zeta \mapsto \zeta^j$. At $j = e$ this character is trivial. A ramified simple module has nontrivial inertia, so its isotypic multiplicity at the middle depth is zero. □

Lemma 6.11 (Faithful-image projectivity). *If V is ramified, then V and $V^\vee$ are projective $\mathbb{F}_2[H_V]$ -modules.*

Proof. We use Clifford theory over an algebraic closure to describe the inertia weights, prove regularity on Sylow-2 weight blocks, and then descend projectivity to $\mathbb{F}_2$.

Let $\overline{\mathbb{F}}_2$ be an algebraic closure, let $I \triangleleft H_V$ be tame inertia, and choose an absolutely irreducible constituent W_0 of $V \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2$. Since $|I|$ is odd, $W_0|_I$ is semisimple. Choose a nontrivial inertia character χ occurring in it, and let H_χ be its stabilizer. Clifford theory expresses W_0 as the induction of a simple H_χ -module on which I acts through χ . The quotient H_χ/I is cyclic. Since χ is H_χ -invariant and $\overline{\mathbb{F}}_2$ contains all roots needed to satisfy the single cyclic relation, χ extends to a one-dimensional character of H_χ . After twisting by that extension, the inducing module factors through the cyclic quotient. Over $\overline{\mathbb{F}}_2$ every simple module for a cyclic group is one-dimensional, and its 2-primary subgroup acts trivially. Thus every χ -weight space in W_0 is one-dimensional.

Let the Frobenius orbit of χ have length $f = 2^a r$ with r odd, and let P be a Sylow 2-subgroup of H_V . The image of P in the cyclic quotient H_V/I is cyclic. Its action on the orbit of χ has 2-power orbit length 2^a . If an element of P stabilizes χ , it acts on the one-dimensional χ -weight space by a scalar of 2-power order, hence trivially because $\overline{\mathbb{F}}_2^\times$ has no nontrivial 2-power torsion. By conjugacy it then acts trivially on all inertia-weight spaces of W_0 , and on every Galois conjugate constituent as well. Faithfulness of the image H_V therefore shows that this stabilizer is trivial. Consequently $|P| = 2^a$, and P acts freely on the weights in blocks of size 2^a .

On each block, transport a nonzero vector around the P -orbit. Because the generator of P has order 2^a , the transported vectors form a regular $\overline{\mathbb{F}}_2[P]$ -basis. Hence the restriction of every absolutely irreducible constituent, and therefore of $V \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2$, is a direct sum of regular $\overline{\mathbb{F}}_2[P]$ -modules. It is projective over $\overline{\mathbb{F}}_2[P]$.

Projectivity descends along the faithfully flat extension $\mathbb{F}_2 \subset \overline{\mathbb{F}}_2$, so $V|_P$ is projective over $\mathbb{F}_2[P]$. The Sylow criterion for modular projectivity now gives projectivity over $\mathbb{F}_2[H_V]$. Replacing every inertia character by its inverse gives the identical Sylow restriction calculation for $V^\vee$, and hence proves projectivity of the dual module. $\square$

Remark 6.12 (Why the faithful image is required). Projectivity is asserted only for the faithful image H_V . A larger tame quotient may contain an additional unramified 2-power subgroup acting trivially on V , and projectivity over that larger group can fail. The base determinant connecting map depends only on H_V ; information involving the extra kernel belongs to the inflated and filtration-one components separated from the base determinant calculation.

Normalized-cochain conventions. From this point through [Lemma 6.15](#), all coefficient modules are left modules and all cochains are normalized inhomogeneous cochains. The differential, cup product, semidirect-product convention, Shapiro map, corestriction, and all changes of sections and transversals are fixed in [Appendix 12](#). In particular, “equal as normalized cochains” means literal equality in the indicated cochain group; when a choice changes a formula, the corresponding normalized coboundary is the one displayed in [Subsection 7](#).

Lemma 6.13 (Universal two-point normalization and the index-two Evens norm). *Let*

$$J = \langle s \rangle \cong C_2, \quad E_2 = \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_s, \quad s(x_1, x_s) = (x_s, x_1),$$

and put

$$q_{\text{hyp}}(x_1, x_s) = x_1 x_s, \quad f_J(x, y) = x_1 y_s, \quad m_1 = 0, \quad m_s(y) = y_1 y_s.$$

Then

$$\kappa_J((x, c), (y, d)) = f_J(x, cy) + m_c(y) \tag{6.37}$$

is a normalized 2-cocycle on $E_2 \rtimes J$. It restricts to the quadratic map q_{hyp} on E_2 and to zero on the zero section J . Its fibre extension is the dihedral group D_8 : the two coordinate lifts are involutions, their commutator is the central involution, and their product has square equal to that involution. Moreover

$$[\kappa_J] = N_{E_2}^{E_2 \rtimes J, \text{Ev}}(e_1^\vee) \quad \text{in } H^2(E_2 \rtimes J, \mathbb{F}_2). \quad (6.38)$$

Now let $N \triangleleft G$ have index 2, identify G/N with J , and let $\alpha \in Z^1(N, \mathbb{F}_2)$. Choose lifts $\tilde{1} = 1$ and $\tilde{s} \in G$, and define the normalized Shapiro cocycle

$$b(\gamma)_u = \alpha\left(\tilde{u}^{-1} \gamma \tilde{\gamma}^{-1} u\right), \quad u \in J. \quad (6.39)$$

This is the degree-one normalized Shapiro map (6.12). The graph pullback of (6.37) is

$$\nu_\alpha(\gamma, \eta) = b(\gamma)_1 b(\eta)_{\tilde{\gamma}^{-1}s} + \varepsilon(\tilde{\gamma}) b(\eta)_1 b(\eta)_s, \quad (6.40)$$

where $\varepsilon(1) = 0$ and $\varepsilon(s) = 1$. Its class is the index-two Evens norm:

$$[\nu_\alpha] = N_N^{G, \text{Ev}}([\alpha]) \in H^2(G, \mathbb{F}_2). \quad (6.41)$$

Equivalently, the normalization is

$$N_N^G(1 + [\alpha]) = 1 + \text{cor}_N^G[\alpha] + N_N^{G, \text{Ev}}([\alpha]).$$

Finally, if $g \in H$ is an involution, $M = \mathbb{F}_2[H]$, $J = \langle g \rangle$, and $\pi : M \rightarrow \mathbb{F}_2[J]$ selects the coordinates indexed by 1 and g , then for every right-coset transversal $\mathcal{R}$,

$$[\kappa_g^\mathcal{R}] = \text{cor}_{M \rtimes J}^{M \rtimes H}(\pi \rtimes 1)^*[\kappa_J]. \quad (6.42)$$

Proof. We verify the universal cocycle and its fibre extension, identify its class with the Evens norm, pull it back to a general index-two inclusion, and then compare the regular-module half-orbit transfer.

Expanding m_s on the two coordinates gives the two identities required in Lemma 6.1: for $v, w \in E_2$,

$$m_s(v + w) + m_s(v) + m_s(w) = f_J(sv, sw) + f_J(v, w), \quad m_s(sv) + m_s(v) = 0.$$

Thus (6.37) is a normalized cocycle. Its square map is $x_1 x_s$, and it vanishes when both fibre coordinates are zero. If $\tilde{e}_1, \tilde{e}_s$ are the coordinate lifts, then $\tilde{e}_1^2 = \tilde{e}_s^2 = 1$, their commutator is the central generator, and $(\tilde{e}_1 \tilde{e}_s)^2$ is that generator; hence the fibre extension is D_8 .

For the swap module,

$$H^1(J, E_2^\vee) = \frac{\ker(1 + s)}{\text{im}(1 + s)} = 0. \quad (6.43)$$

The low-degree exact sequence (6.22) therefore says that a class on $E_2 \rtimes J$ is determined by its restriction to E_2 and its restriction to the zero section. Mackey's formula for the standard Evens norm gives

$$\text{res}_{E_2} N_{E_2}^{E_2 \rtimes J, \text{Ev}}(e_1^\vee) = e_1^\vee e_s^\vee, \quad \text{res}_J N_{E_2}^{E_2 \rtimes J, \text{Ev}}(e_1^\vee) = 0.$$

These are exactly the two restrictions of $[\kappa_J]$, proving (6.38). Equivalently, this class is the degree-two component of the relative Stiefel–Whitney class $w(\text{Ind } \ell)/w(\text{Ind } 1)$ under the Evens–Kahn normalization; see [9], [10], [11].

For the general index-two inclusion, the graph

$$\Phi_\alpha : G \longrightarrow E_2 \rtimes J, \quad \gamma \longmapsto (b(\gamma), \bar{\gamma})$$

has inverse image of E_2 equal to N , and the pullback of $e_1^\vee$ to N is α . Naturality of the Evens norm in this pullback square and (6.38) give (6.41). Direct substitution of (6.39) into (6.37) gives (6.40) term by term. This is also the degree-two term in Evens's normalized bar construction [9, §§4–5]. If $\tilde{s}$ is replaced by $\tilde{s}a$ with $a \in N$, then (2.25) gives the exact change of the Shapiro cocycle, and (3.36) gives the resulting normalized 1-cochain whose boundary is the change in (6.40). Thus the lift dependence is controlled at cochain level, not only in cohomology.

For (6.42), Mackey restriction of the right side to M is

$$\sum_{u \in H/J} x_u x_{ug},$$

and its zero-section restriction is zero. The explicit class $[\kappa_g^{\mathcal{R}}]$ has the same two restrictions. Moreover $H^1(H, M^\vee) = 0$ by Shapiro's lemma, since $M^\vee$ is coinduced. The same low-degree uniqueness argument proves equality. More precisely, (3.30) identifies the two representatives as normalized cochains for the chosen transversal, and (3.32) displays the normalized 1-cochain whose boundary is the change under an arbitrary new transversal. $\square$

Lemma 6.14 (Regular-module realization of the base local connecting map). *With notation as in Lemma 6.3, the base local determinant form satisfies, for every $x \in H^1(\mathbb{Q}_2, V)$,*

$$Q_{\text{loc},q}^0(x) = Q_{\text{loc},q_W}^0(i_*x). \quad (6.44)$$

Under Kummer theory and the normalized Shapiro cochain map, the right side is obtained by applying the orbit operations of Lemma 6.15 to the scalar Shapiro coordinates. Thus the regular-module computation evaluates the actual base equivariant central class κ_q^0 , including the m_c -terms; it is not merely a polynomial with the same polarization.

Proof. The equality of classes is the pullback identity $\kappa_q^0 = (i \rtimes 1)^* \kappa_{q_W}^0$ from (6.19). Pulling that identity back along a Galois graph cocycle gives equality of normalized central 2-cocycles over $G_{\mathbb{Q}_2}$, hence (6.44) after applying the local invariant map.

For the second assertion, use the explicit representatives in (6.17), (6.18), and (6.15). Direct substitution of the normalized Shapiro cocycle gives the square and free-orbit corestriction formulas. For a half-orbit, the correction term m_c^g records exactly whether the action on a right coset of $\langle g \rangle$ preserves or reverses the chosen orientation of its two coordinates. Thus the graph pullback is the normalized corestriction of the two-point graph cocycle (6.40); by Lemma 6.13 that two-point class is the index-two Evens norm. The full cochain calculation is written out in Lemma 6.15. $\square$

Lemma 6.15 (Quadratic orbit-stabilizer Shapiro). *Let $W = \mathbb{F}_2[H]^N$ have coordinates $X_{r,h}$ with H acting by left translation on h . If K/F is the H -extension and $\alpha_r \in H^1(K, \mathbb{F}_2)$ is the scalar Shapiro coordinate, then the orbit classes of Lemma 6.3 evaluate as follows. For $g \in H$, choose any lift $\tilde{g} \in G_F$ and use the convention*

$$(g\alpha_s)(n) = \alpha_s(\tilde{g}^{-1}n\tilde{g}) \quad (n \in G_K).$$

This is independent of the chosen lift and is the convention fixed in (1.13):

$$S_r \longmapsto \text{cor}_{K/F}(\alpha_r^2), \quad (6.45)$$

$$C_{r,s,g} \mapsto \text{cor}_{K/F}(\alpha_r \smile g\alpha_s), \quad (6.46)$$

$$E_{r,g} \mapsto \text{cor}_{K_0/F} N_{K/K_0}^{\text{Ev}}(\alpha_r), \quad K_0 = K^{\langle g \rangle}. \quad (6.47)$$

Proof. We treat square orbits, free orbits, and involution-stabilized orbits in that order, recording at each stage whether the identity is literal at cochain level or only after passage to cohomology.

Choose lifts $\tilde{h} \in G_F$ of $h \in H$, and write $\bar{\gamma}$ for the image of $\gamma \in G_F$ in H . With the left-regular convention used here, the normalized Shapiro map is

$$b_r(\gamma)_h = \text{Sh}(\alpha_r)(\gamma)_h = \alpha_r(\tilde{h}^{-1} \gamma \widetilde{\bar{\gamma}^{-1} h}).$$

For scalar cocycles α, β on G_K , direct substitution into the normalized bar corestriction gives

$$\sum_{h \in H} \text{Sh}(\alpha)(\gamma)_h \text{Sh}(\beta)(\eta)_{\bar{\gamma}^{-1}h} = \text{cor}_{K/F}(\alpha \smile \beta)(\gamma, \eta). \quad (6.48)$$

The factor set (6.17) therefore gives (6.45). For the free orbit, the coordinate appearing in the monomial is the right shift $R_g \text{Sh}(\alpha_s)$. By (15) this differs from $\text{Sh}(g\alpha_s)$ by the explicit coboundary $dq_{\alpha_s, g}$, and (16) displays the resulting normalized boundary after taking the cup product and trace. Therefore (6.46) holds in cohomology; it is a literal cochain identity whenever the chosen section is right- g -equivariant.

Now let $J = \langle g \rangle$, let $\mathcal{R}$ be right-coset representatives for H/J , and put $K_0 = K^J$. For $c \in H$ and $u \in \mathcal{R}$, use the notation

$$c^{-1}u = \pi_c(u)g^{\epsilon_c(u)}$$

of Lemma 6.2. Let $\rho_0 : G_{K_0} \twoheadrightarrow J$ be the quotient map. The graph pullback of the explicit half-orbit cocycle (6.15) is

$$\begin{aligned} & \sum_{u \in \mathcal{R}} b_r(\gamma)_u b_r(\eta)_{\bar{\gamma}^{-1}ug} \\ & + \sum_{u \in \mathcal{R}} \epsilon_{\bar{\gamma}}(u) b_r(\eta)_{\pi_{\bar{\gamma}}(u)} b_r(\eta)_{\pi_{\bar{\gamma}}(u)g}. \end{aligned} \quad (6.49)$$

For completeness, choose the lifts compatibly so that $\widetilde{ug} = \tilde{u} \tilde{g}$ for $u \in \mathcal{R}$, and put

$$\ell_u(\gamma) = \tilde{u}^{-1} \gamma \widetilde{\pi_{\bar{\gamma}}(u)} \in G_{K_0}.$$

Its image in J is $g^{\epsilon_{\bar{\gamma}}(u)}$. In these conventions the normalized corestriction of a 2-cochain ν on G_{K_0} is

$$(\text{cor}_{K_0/F} \nu)(\gamma, \eta) = \sum_{u \in \mathcal{R}} \nu(\ell_u(\gamma), \ell_{\pi_{\bar{\gamma}}(u)}(\eta)). \quad (6.50)$$

Let $b_J = \text{Sh}_{K/K_0}(\alpha_r)$. Compatibility of the transversals gives

$$b_J(\ell_u(\gamma))_1 = b_r(\gamma)_u$$

and, for $j \in \{1, g\}$,

$$b_J(\ell_{\pi_{\bar{\gamma}}(u)}(\eta))_j = b_r(\eta)_{\pi_{\bar{\gamma}}(u)j}.$$

Substituting the explicit two-point cocycle (6.40) into (6.50) gives exactly (6.49): its first summand is the oriented factor-set term, and its second summand is the orientation-reversal correction. Consequently

$$(b_r, \rho)^* \kappa_g^{\mathcal{R}} = \text{cor}_{K_0/F}((\text{Sh}_{K/K_0} \alpha_r, \rho_0)^* \kappa_J) \quad (6.51)$$

as normalized cochains. By Lemma 6.13, the inner graph class is $N_{G_K}^{G_{K_0}, \text{Ev}}(\alpha_r)$, which is denoted $N_{K/K_0}^{\text{Ev}}(\alpha_r)$ in field notation. Equation (6.47) follows. With the compatible nested transversals just chosen, (6.51) is literal equality of normalized cochains. For arbitrary choices, (.32) and (.34) give the two explicit normalized 1-cochains whose boundaries account for the change. Hence the formulas evaluate the actual orbit classes used in $\kappa_{q_W}^0$, including all m_c -terms, without an unspecified convention-dependent ambiguity. $\square$

Lemma 6.16 (Deep-unit Evens norm). *Let L/k be an unramified quadratic extension of finite dyadic local fields and put $e = v_k(2)$. Then*

$$N_{L/k}^{\text{Ev}}(\alpha) = 0 \quad (\alpha \in U_{e+1}(L) \subset L^\times / L^{\times 2}). \quad (6.52)$$

Proof. We translate the Evens norm into the degree-two Evens–Kahn identity, reduce the trace forms to Hilbert symbols, and then cancel the remaining terms by norm and Steinberg relations.

By Lemma 6.13, the Evens norm is represented by the explicit two-point graph cocycle (6.40). The index-two Evens–Kahn formula identifies this normalization with the degree-two term in the relative Stiefel–Whitney identity [10], [11]

$$w(\text{Tr}_{L/k} \langle a \rangle) = w(\text{Tr}_{L/k} \langle 1 \rangle)(1 + \text{cor}_{L/k}[a] + N_{L/k}^{\text{Ev}}([a])). \quad (6.53)$$

Take a class in $U_{e+1}(L)$ and choose from the outset a representative

$$a = 1 + 2b, \quad b \in \mathfrak{p}_L.$$

Write $L = k(\sqrt{d})$, where d is the unramified quadratic square class, and write

$$a = u + v\sqrt{d}, \quad n = N_{L/k}(a) = u^2 - dv^2.$$

If

$$t = \text{Tr}_{L/k}(b), \quad m = N_{L/k}(b),$$

then

$$u = 1 + t \in \mathcal{O}_k^\times, \quad n = 1 + 2t + 4m. \quad (6.54)$$

In particular the divisions by u below are legitimate. The trace forms diagonalize as

$$\text{Tr}_{L/k} \langle a \rangle \simeq \left\langle 2u, \frac{2dn}{u} \right\rangle, \quad \text{Tr}_{L/k} \langle 1 \rangle \simeq \langle 2, 2d \rangle.$$

Taking degree two in (6.53) first gives

$$N_{L/k}^{\text{Ev}}([a]) = (2u, 2dn/u) + (2, 2d) + (d, n).$$

We now keep every Hilbert-symbol cancellation visible. Bilinearity and $(x, x) = (x, -1)$ give

$$(2u, 2dn/u) = (2u, 2) + (2u, d) + (2u, n) + (2u, u),$$

$$\begin{aligned}(2u, 2) + (2u, u) &= (2u, 2u) = (2u, -1), \\ (2, 2d) &= (2, -1) + (2, d).\end{aligned}$$

Consequently

$$\begin{aligned}N_{L/k}^{\text{Ev}}([a]) &= (u, -1) + (u, d) + (2u, n) + (d, n) \\ &= (u, -d) + (2du, n).\end{aligned}$$

Thus

$$N_{L/k}^{\text{Ev}}([a]) = (u, -d) + (2du, n). \quad (6.55)$$

Because L/k is unramified and $b \in \mathfrak{p}_L$, one has $v_k(m) \geq 2$. From (6.54),

$$\frac{n}{2u-1} = 1 + \frac{4m}{2u-1} \in U_{2e+2}(k). \quad (6.56)$$

Here $2u-1$ is a unit and $v_k(4m) \geq 2e+2 > 2v_k(2)$. Hensel's lemma, applied to $X^2 - n/(2u-1)$ at $X=1$, therefore gives $U_{2e+2}(k) \subset k^{\times 2}$. Thus n and $2u-1$ have the same square class.

Every unit of k is a norm from its unramified quadratic extension, and $n = N_{L/k}(a)$. Hence

$$(u, d) = 0, \quad (d, n) = 0.$$

Formula (6.55) becomes

$$(u, -1) + (2u, 2u-1).$$

The Steinberg relation gives $(2u, 2u-1) = (2u, -1)$. Finally $(2, -1) = 0$: if $i \in k$, then -1 is a square; otherwise $2 = N_{k(i)/k}(1+i)$. Consequently $(2u, -1) = (u, -1)$, and the two terms cancel. $\square$

Lemma 6.17 (The deep half is totally singular). *Assume V is ramified and put*

$$X_+ = \text{Hom}_{H_V}(V^\vee, U_{e+1}) \subset H^1(\mathbb{Q}_2, V).$$

Then

$$\dim X_+ = \frac{1}{2} \dim H^1(\mathbb{Q}_2, V), \quad Q_{\text{loc}}^0|_{X_+} = 0.$$

Proof. We first count the contributing unit-filtration layers and then prove vanishing on the deep half by checking square, free, and involution orbits.

By Hochschild–Serre and Kummer theory,

$$H^1(\mathbb{Q}_2, V) \cong \text{Hom}_{H_V}(V^\vee, M_K),$$

and $\text{Hom}_{H_V}(V^\vee, -)$ is exact on the unit filtration. Hilbert duality pairs the depth- j and depth- $(2e-j)$ graded pieces, and self-duality of V gives equal V -isotypic multiplicities in each pair. The only unpaired depth is $j=e$; by Lemma 6.10 it has trivial inertia and contains no ramified copy of V . Hence exactly one member of every contributing pair lies above e , proving the dimension statement.

By Lemma 6.14, compute the base connecting map after an H_V -split embedding into a regular module, and apply Lemma 6.15 to the complete invariant monomial expansion of $q \circ p$. For a class in X_+ every scalar coordinate α_r lies in $U_{e+1}(K)$. Free orbits vanish by (6.36). Square orbits vanish because $\alpha_r^2 = \alpha_r \smile (-1)$ and $-1 \in U_e$. For an involution orbit, tame inertia has odd order, so K/K_0 in (6.47) is unramified quadratic. The depth parameter is unchanged in an unramified subextension, and Lemma 6.16 makes the Evens norm vanish before corestriction. Every orbit contribution is therefore zero. $\square$

6.4 Comparison of the Gauss signs

Proposition 6.18 (Dyadic base determinant theorem). *If $d = \dim V$, then*

$$\#(Q_{\text{loc}}^0)^{-1}(0) = \begin{cases} 2^{d-1} - 2^{d/2-1}, & V \text{ unramified,} \\ 2^{d-1} + 2^{d/2-1}, & V \text{ ramified.} \end{cases} \quad (6.57)$$

Equivalently, the base local Gauss sum is negative in the unramified case and positive in the ramified case. The candidate base form Q_A^0 has the same Gauss sum by Proposition 6.9.

Proof. The ramified and unramified cases are treated separately: the first follows from the half-dimensional totally singular subspace, while the second is determined by the invariant Hermitian-line classification.

In the ramified case, local Euler–Poincaré and Tate duality give $\dim H^1(\mathbb{Q}_2, V) = d$. By Lemma 6.17, the nonsingular base local quadratic space contains a totally singular subspace of dimension $d/2$, so it is hyperbolic.

In the unramified case, the faithful image is cyclic of odd order. Put $F_V = \text{End}_{H_V}(V)$. Self-duality equips $F_V \cong \mathbb{F}_{2^{2m}}$ with the involution $x \mapsto x^* = x^{2^m}$ and fixed field $F_{V,0} \cong \mathbb{F}_{2^m}$. Inflation–restriction and the local Euler characteristic identify $H^1(\mathbb{Q}_2, V)$ with a one-dimensional F_V -space. The norm-one group

$$\mathcal{U} = \{u \in F_V^\times : uu^* = 1\}$$

preserves the polar form of q . For $u \in \mathcal{U}$, the function $x \mapsto q(ux) + q(x)$ is linear and H_V -invariant. Since $(V^\vee)^{H_V} = 0$, it vanishes; hence every $u \in \mathcal{U}$ is an H_V -equivariant q -isometry. By (6.21), $(u \rtimes 1)^*[\kappa_q^0] = [\kappa_q^0]$. Functoriality of graph pullback therefore gives

$$Q_{\text{loc}}^0(u_*x) = Q_{\text{loc}}^0(x) \quad (u \in \mathcal{U}).$$

Thus Lemma 6.7 applies to the base local form, and it is

$$Q(x) = \text{Tr}_{F_{V,0}/\mathbb{F}_2}(axx^*), \quad a \in F_{V,0}^\times.$$

The norm map $F_V^\times \rightarrow F_{V,0}^\times$ has fibres of size $2^m + 1$. Among the nonzero elements of $F_{V,0}$, exactly $2^{m-1} - 1$ have trace 0. Hence

$$\#Q^{-1}(0) = 1 + (2^m + 1)(2^{m-1} - 1) = 2^{2m-1} - 2^{m-1},$$

which is the minus-type number in (6.57). □

Thus the candidate and local *base* determinant problems have the same Gauss sum. For an arbitrary determinant pushout class, the edge term γ_κ and the inflated scalar term δ_κ remain in the obstruction formula; their behavior under transgression and shear is described in Subsection 6.5.

Corollary 6.19 (Complete source interface). *For every finite lower quotient, every lower exact-image map, and every elementary characteristic-2 coefficient module, the two sources have:*

1. *the same obstruction-space dimension and the same number of lifts when the obstruction vanishes;*
2. *natural perfect pairings for all short exact coefficient sequences, with adjoint connecting homomorphisms;*

3. the same scalar cup–Bockstein form for the trivial module;
4. for every nonzero invariant nonsingular quadratic form on a simple self-dual head, the same base determinant Gauss sum for the class κ_q^0 .

Together with the marked maximal pro-2 identification in [Proposition 3.10](#), these are the source-side inputs supplied to the target induction. This corollary is not by itself a determinacy statement: equality of base Gauss sums is used only after Fourier expansion. For a general pushout class

$$\kappa = \kappa_q^0 + \Gamma_{\gamma_\kappa} + \inf \delta_\kappa,$$

the terms γ_κ and δ_κ enter only through the scalar classes obtained by completing the square in [Proposition 8.8](#). The corresponding signed sums are expressed as exact-image counts in central double covers of the lower target.

Proof. Items (i)–(iii) are [Proposition 5.15](#), [Proposition 5.16](#), [Corollary 5.17](#) and [\(3.18\)](#); item (iv) is [Proposition 6.9](#), [Proposition 6.18](#). The later operations are restrictions, pushouts, exact-image partitions, finite Fourier transforms, and central-cover constructions internal to the finite target. The determinacy of those operations is isolated in [Theorem 8.17](#) and then executed in [Section 9](#); the central-cover identity [\(8.22\)](#) supplies the remaining affine information. $\square$

6.5 Affine terms: transgression and shear

Lemma 6.20 (No square-only invariant). *Let V be a nontrivial simple $\mathbb{F}_2[C]$ -module. Every nonzero element of $H^0(C, \text{Sym}^2 V^\vee)$ has nonzero, hence nonsingular, polar form.*

Proof. The zero-polar submodule is the Frobenius-square copy of $V^\vee$, whose fixed subspace is zero. The radical of a nonzero invariant polar form is a submodule of the simple module V . $\square$

Lemma 6.21 (Determinant transgression relative to the fixed equivariant class). *Let q be a nonsingular C -invariant quadratic form on V , and assume that a zero-section-normalized equivariant class*

$$\kappa_q^0 \in H^2(V \rtimes C, \mathbb{F}_2)$$

restricting to q on V has been fixed. Let

$$1 \rightarrow V \rightarrow G_\eta \rightarrow C \rightarrow 1$$

have extension class $\eta \in H^2(C, V)$. Relative to the fixed equivariant lift κ_q^0 , the obstruction to extending the fibre class over G_η is

$$d_2(q) = b_{q*}^\flat \eta \in H^2(C, V^\vee). \tag{6.58}$$

Consequently, if q is the restriction of an actual class in $H^2(G_\eta, \mathbb{F}_2)$, then $\eta = 0$ and $G_\eta \cong V \rtimes C$ over C .

Proof. We choose sections for the extension, compare the induced automorphisms of the fibre extension, and identify their discrepancy with the transgressed class $b_{q*}^\flat(\eta)$.

Let

$$1 \longrightarrow \mathbb{F}_2 \longrightarrow E_q \longrightarrow V \longrightarrow 1$$

be the central extension represented by $\kappa_q^0|_V$. Central automorphisms of E_q inducing the identity on V are canonically $V^\vee$. If $\widehat{v}$ is a lift of $v \in V$, inner conjugation by $\widehat{v}$ is the central automorphism

$$w \mapsto b_q(v, w) = b_q^b(v)(w). \quad (6.59)$$

Choose the zero-section-normalized representative of κ_q^0 . Its preimage of C is split, so conjugation by the chosen lifts of $c \in C$ supplies coherent automorphisms

$$\alpha_c \in \text{Aut}(E_q), \quad \alpha_c \alpha_d = \alpha_{cd},$$

lifting the prescribed C -action on V .

Choose a normalized section $s : C \rightarrow G_\eta$ and write

$$s(c)s(d) = \eta(c, d)s(cd). \quad (6.60)$$

In a putative central extension of G_η restricting to E_q , lifts of the section elements would satisfy

$$\widehat{s}(c)\widehat{s}(d) = z(c, d)\widehat{\eta(c, d)}\widehat{s}(cd)$$

for a scalar factor $z(c, d) \in \mathbb{F}_2$. Compare the induced automorphisms of E_q . The left side induces $\alpha_c \alpha_d = \alpha_{cd}$, whereas the right side induces

$$\text{Int}(\widehat{\eta(c, d)})\alpha_{cd}.$$

By (6.59), their discrepancy is the $V^\vee$ -valued normalized 2-cocycle

$$(c, d) \mapsto b_q^b(\eta(c, d)). \quad (6.61)$$

Changing the section s changes η by a coboundary and changes (6.61) by the corresponding $V^\vee$ -coboundary. Changing the coherent lifts α_c by central automorphisms $\gamma(c) \in V^\vee$ changes the discrepancy by $d\gamma$. Hence the cohomology class of the obstruction is exactly $b_{q*}^b[\eta]$. This is the $d_2^{0,2}$ differential after the intrinsic equivariance obstruction has been trivialized by the fixed split class κ_q^0 , proving (6.58).

If an actual class on G_η restricts to q , the obstruction vanishes. Since q is nonsingular, $b_q^b : V \xrightarrow{\sim} V^\vee$ is an isomorphism of C -modules. Therefore $[\eta] = 0$, so the extension splits over C . $\square$

For a split group $V \rtimes C$, fix the base determinant class κ_q^0 of Lemma 6.3. After replacing a representative by a cohomologous normalized cocycle and changing the normalized section, every central class with restriction q admits the following equality of normalized 2-cochains:

$$\kappa = \kappa_q^0 + \Gamma_\gamma + \inf_C^{V \rtimes C} \delta, \quad (6.62)$$

where $\gamma \in Z^1(C, V^\vee)$, $\delta \in Z^2(C, \mathbb{F}_2)$, and Γ_γ is given by (6.6).

Lemma 6.22 (Marking-preserving shear of the difference data). *Let $a \in Z^1(C, V)$ and define*

$$s_a(v, c) = (v + a(c), c).$$

If $\kappa = \kappa_q^0 + \Gamma_\gamma + \inf \delta$, then, in cohomology,

$$s_a^* \kappa = \kappa_q^0 + \Gamma_{\gamma + b_q^b a} + \inf(\delta + \Theta_q^0(a) + \gamma \smile a), \quad (6.63)$$

where

$$\Theta_q^0(a) = (a, \text{id}_C)^* \kappa_q^0 \in Z^2(C, \mathbb{F}_2) \quad (6.64)$$

and

$$(\gamma \smile a)(c, d) = \gamma(c)(ca(d)). \quad (6.65)$$

In particular, if $q \neq 0$, a unique cohomology class a kills the edge γ , and the scalar phase then becomes $\delta + \Theta_q^0(a) + \gamma \smile a$.

Proof. The cocycle identity for a is exactly the condition that s_a be an automorphism over C . Substitute $v + a(c)$ and $w + a(d)$ into the normalized representative (6.62). The terms linear in v or w change the edge by the polar map $b_q^b a$. The term containing only the base class and the graph a is $\Theta_q^0(a)$. The old edge also contributes the scalar term $\gamma(c)(ca(d))$, namely (6.65). All remaining changes are normalized coboundaries. Since b_q^b is an isomorphism, the edge can be killed uniquely in cohomology.

The shear fixes the quotient C . A marked subgroup containing V is the full inverse image of its image in C , so it is preserved. Scalar decorations kill the nontrivial simple module V , and are therefore preserved as well. $\square$

7 A minimal non-scalar module layer in the wild kernel

Given a finite target whose marked 2-kernel has a nontrivial composition factor, this section isolates the first such factor and the minimal normal subgroup that realizes it. The resulting quotient tower separates the elementary lifting stage from the final Frattini obstruction and shows that every nonzero scalar pushout induces a nonsingular quadratic form on the simple head.

Let $\mathcal{Y} = (Y, L_Y, \pi_Y, \theta_Y)$ be a boundary-framed target, so that $L_Y = \ker(\pi_Y)$ is a finite 2-group and $H = Y/L_Y$. For a finite 2-group K , write

$$\Phi(K) = K^2[K, K]$$

for its Frattini subgroup. A chief factor is scalar when the tame quotient acts on it trivially. Thus, formally, a chief factor of L_Y is called *scalar* when it is the trivial $\mathbb{F}_2[H]$ -module.

Assume that L_Y has a non-scalar chief factor. Choose Y -normal subgroups

$$S \triangleleft P \leq L_Y, \quad P/S \cong V,$$

where all chief factors below S are scalar and V is the first non-scalar simple factor. Choose $K \triangleleft Y$, $K \leq P$, minimal subject to $KS = P$, and set

$$R = \Phi(K), \quad M = K/R, \quad T = (K \cap S)/R,$$

$$B = Y/R, \quad C = Y/K, \quad H = Y/L_Y, \quad H_V = \text{im}(C \rightarrow \text{GL}(V)).$$

The quotient tower and the elementary module layer are

$$Y \twoheadrightarrow B = Y/R \twoheadrightarrow C = Y/K \twoheadrightarrow H = Y/L_Y, \quad 0 \longrightarrow T \longrightarrow M \longrightarrow V \longrightarrow 0. \quad (7.1)$$

Write $L_B = L_Y/R$ and $L_C = L_Y/K$. The four quotient levels used below are

Level	Role
H	fixed tame quotient and boundary frame
C	lower target for elementary M -lifting, connecting maps, and scalar phase classes
B	quotient on which the final R -valued obstruction and its scalar pushouts are evaluated
H_V	faithful image used only for the pure quadratic-form calculation on the simple head V

Lemma 7.1 (Simple head). *There is an exact sequence of $\mathbb{F}_2[C]$ -modules*

$$0 \rightarrow T \rightarrow M \rightarrow V \rightarrow 0,$$

and

$$T = \text{rad}_{\mathbb{F}_2[C]} M, \quad M/T \cong V, \quad (M^\vee)^C = 0.$$

Proof. Since $K/(K \cap S) \cong V$ is elementary abelian, $R \leq K \cap S$. The displayed sequence follows. Since the quotient is simple, $\text{rad } M \leq T$. If a maximal submodule $X < M$ failed to contain T , then $X + T = M$, so X would map onto V . Writing K_X for the inverse image of X in K , the subgroup K_X would be proper and Y -normal while still satisfying $K_X S = P$, contradicting minimality. Thus every maximal submodule contains T . The assertion about dual invariants follows because a nonzero map $M \rightarrow \mathbb{F}_2$ would give a trivial simple quotient, whereas the unique simple head is V . $\square$

Lemma 7.2 (Frattini-centralizer collapse). *The subgroup R is central elementary abelian in K , and $K^4 = 1$.*

Proof. We first prove that K centralizes its Frattini subgroup, then show that the fourth-power homomorphism vanishes, and finally identify R as central and elementary abelian.

Let H_2 be the maximal 2-quotient of the finite tame group H , let $N = \ker(H \rightarrow H_2)$, and choose an odd Hall lift $\tilde{N}$ in the preimage of N in Y . Along a chief series for S , the action of $\tilde{N}$ on every factor is trivial; coprime action gives

$$[S, \tilde{N}] = 1.$$

Since $R \leq K \cap S$, $\tilde{N}$ centralizes R .

Put $\mathcal{C}_R = C_K(R)$. It is characteristic in K and hence normal in Y . The three-subgroup lemma gives $[K, \tilde{N}] \leq \mathcal{C}_R$. The group N acts nontrivially on V : otherwise the action would factor through the 2-group H/N , whose only simple module in characteristic 2 is trivial. Hence $[V, N] = V$, so

$$[K, \tilde{N}]S = P.$$

Thus $\mathcal{C}_R S = P$, and minimality of K gives $\mathcal{C}_R = K$. Therefore $R \leq Z(K)$.

Since K/R is elementary abelian, squares lie in R and commutators have order at most 2. The fourth-power map

$$f : K \rightarrow R, \quad f(k) = k^4,$$

is a Y -equivariant homomorphism. The subgroup $\tilde{N}$ fixes R , so $f([k, n]) = 1$ for $k \in K$, $n \in \tilde{N}$. The Y -normal closure of $[K, \tilde{N}]$ still maps onto V modulo S , hence equals K by minimality. Thus $f = 1$ and $K^4 = 1$. Finally

$$R = \Phi(K) = K^2[K, K]$$

is central of exponent 2. $\square$

Lemma 7.3 (Decorations vanish on the block). *Every homomorphism from Y to an elementary abelian 2-group vanishes on K .*

Proof. It kills $R = K^2[K, K]$ and therefore induces a C -module map $M \rightarrow E$. By Lemma 7.1, $\text{Hom}_C(M, \mathbb{F}_2) = 0$, so this map is zero. $\square$

7.1 The determinant pushout on the simple head

Put

$$\mathcal{X}_R = (R^\vee)^B = (R^\vee)^C.$$

For $0 \neq \lambda \in \mathcal{X}_R$, the pushout $K_\lambda = K / \ker \lambda$ is a central extension

$$1 \rightarrow \mathbb{F}_2 \rightarrow K_\lambda \rightarrow M \rightarrow 1.$$

Let $q_\lambda : M \rightarrow \mathbb{F}_2$ be its square map and b_λ its polar form.

Proposition 7.4 (Simple-head determinant). *For every $0 \neq \lambda \in \mathcal{X}_R$,*

$$q_\lambda|_T = 0, \quad b_\lambda(T, M) = 0.$$

Hence q_λ descends to a nonzero nonsingular C -invariant quadratic form

$$\bar{q}_\lambda : V \rightarrow \mathbb{F}_2.$$

Proof. We identify the polar radical, rule out a nonzero restriction to T , and then prove that the descended form on V is nonzero and nonsingular.

The adjoint $b_\lambda^\flat : M \rightarrow M^\vee$ is C -linear. Since T has only trivial composition factors while $\text{soc}(M^\vee) \cong V^\vee$ is nontrivial simple, $\text{Hom}_C(T, M^\vee) = 0$. Thus $b_\lambda(T, M) = 0$.

The restriction $q_\lambda|_T$ is therefore a C -invariant linear functional. Suppose it is nonzero. Quotienting T by its kernel yields a C -module extension

$$0 \rightarrow \mathbb{F}_2 t \rightarrow \bar{M} \rightarrow V \rightarrow 0,$$

with t in the polar radical and $q_\lambda(t) = 1$. The kernel of $C \rightarrow H_V$ acts on $\bar{M}$ by possible shears

$$m \mapsto m + f(\bar{m})t.$$

Quadratic invariance and $q_\lambda(t) = 1$ force every such f to vanish, so the extension factors through H_V . If V is ramified, its odd inertia subgroup has no fixed vectors and $H^*(H_V, V^\vee) = 0$ by Hochschild–Serre. If V is unramified, H_V is cyclic of odd order and the same vanishing follows from Maschke’s theorem. Hence

$$H^1(H_V, V^\vee) = 0,$$

so the extension splits. Projection to the scalar summand gives a nonzero C -map $M \rightarrow \mathbb{F}_2$, contradicting $(M^\vee)^C = 0$. Therefore $q_\lambda|_T = 0$.

The descended form is nonzero. Otherwise K_λ would be elementary abelian, whereas

$$\Phi(K_\lambda) = R / \ker \lambda \neq 1.$$

Its polar radical is a submodule of the simple module V ; if the radical were all of V , the form would be invariant linear, impossible. Thus it is nonsingular. $\square$

8 Central covers, affine fibres, and Fourier inversion

Using the target tower of [Section 7](#) and the source comparison from [Section 5](#), [Section 6](#), this section converts the lifting count into scalar Fourier coefficients and exact-image counts for smaller targets. The result is the closed recursion of [Theorem 8.17](#).

Retain the tower and module sequence [\(7.1\)](#).

The induction repeatedly separates maps according to the subgroup they generate. The notation below retains only maps whose image is exactly the lower target.

Definition 8.1 (Exact-image lower maps). For a boundary-framed lower target C , let $X_\Gamma(C)$ denote the finite set of boundary-framed exact-image epimorphisms $\rho : \Gamma \twoheadrightarrow C$ at the current lower stage. All sums over $X_\Gamma(C)$ are over exact-image lower maps; proper-image contributions are subtracted separately by induction.

Before the transform can be compared across the two sources, the scalar characters entering it must agree; they do, with the same count on both sides.

Lemma 8.2 (Common scalar character group). *For both sources $\Gamma \in \{\Gamma_A, G_{\mathbb{Q}_2}\}$ one has*

$$|\mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{F}_2)| = 8.$$

Moreover scalar twisting by any such character preserves the boundary-framed condition for the central double covers used below.

Proof. For the candidate, every map to $\mathbb{F}_2$ kills τ because the tame relation forces the tame inertia generator to have odd order in finite quotients; hence it factors through the maximal pro-2 quotient Π . For the local source, the same assertion follows from the characteristic wild subgroup and the common marked maximal pro-2 quotient. The quotient Π has three mod-2 generators, represented by σ, x_0, x_1 , and one Demushkin relator with no linear mod-2 part, so

$$H^1(\Pi, \mathbb{F}_2) \cong \mathbb{F}_2^3.$$

Thus the scalar character group has order $2^3 = 8$ for both sources. The central covers used in the induction have central kernel killed by the fixed boundary and decoration, so multiplying a lift by a scalar character changes only the central coordinate and leaves the boundary-framed data unchanged. $\square$

Lemma 8.3 (Central-cover exact-image transform). *Let $\mathcal{Y} = (Y, L_Y, \pi_Y, \theta_Y)$ be boundary-framed and let*

$$p : \tilde{Y} \twoheadrightarrow Y$$

be a central double cover. Give it the pulled-back structure

$$\tilde{L} = p^{-1}(L_Y), \quad \tilde{\pi} = \pi_Y p, \quad \tilde{\theta} = \theta_Y p,$$

and assume the central kernel lies in $\ker(\tilde{\pi}, \tilde{\theta})$. Fix an exact-image subgroup $J \leq Y$ projecting onto H . If $u_\Gamma^\beta(p, J)$ counts boundary-framed exact-image maps to J whose pullback cover is split, then

$$8u_\Gamma^\beta(p, J) = \sum_{\substack{\tilde{J} \leq p^{-1}(J) \\ p(\tilde{J})=J}} e_\Gamma^\beta(\tilde{J}, \tilde{J} \cap \tilde{L}, \tilde{\pi}|_{\tilde{J}}, \tilde{\theta}|_{\tilde{J}}). \quad (8.1)$$

Every term on the right is therefore an ordinary exact-image object in the same global boundary-framed category.

Proof. An unobstructed map $f : \Gamma \twoheadrightarrow J$ has a lift, and the set of all lifts is a torsor under $\text{Hom}(\Gamma, \mathbb{F}_2)$. By Lemma 8.2, this group has order 8 for both sources, and scalar twisting preserves the boundary-framed condition. Twisting a lift changes only the new central coordinate. Because that coordinate lies in the kernel of both the tame marking and the decoration, every twist still satisfies the identical boundary equation. The exact image of a lift is a subgroup $\tilde{J} \leq p^{-1}(J)$ mapping onto J , and every such exact-image lift occurs in precisely one summand. This is a literal partition of the eight lifts, proving (8.1); it also proves exact-image and decoration compatibility without an equidistribution assumption. $\square$

The transform is inverted by a finite Fourier average.

Lemma 8.4 (Fourier inversion). *Let Λ be the character group of a finite $\mathbb{F}_2$ -obstruction space and let $o : X \rightarrow \Lambda^\vee$. If*

$$m_\lambda = \#\{x \in X : \langle \lambda, o(x) \rangle = 0\},$$

then

$$\#\{x : o(x) = 0\} = \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} (2m_\lambda - |X|). \quad (8.2)$$

Proof. Use $1_{o=0} = |\Lambda|^{-1} \sum_{\lambda \in \Lambda} (-1)^{\langle \lambda, o \rangle}$. $\square$

Lemma 8.5 (Constrained quadratic Gauss transform). *Let W, E be finite $\mathbb{F}_2$ -vector spaces, let $L : W \twoheadrightarrow E$, and let $Q : W \rightarrow \mathbb{F}_2$ be nonsingular with polar form b_Q . For $\kappa \in E$ and $\epsilon \in \mathbb{F}_2$, put*

$$N(\kappa, \epsilon) = \#\{x \in W : Lx = \kappa, Q(x) = \epsilon\}.$$

For $\chi \in E^\vee$, let a_χ be uniquely determined by

$$b_Q(a_\chi, x) = \chi(Lx).$$

Then

$$N(\kappa, \epsilon) = \frac{1}{2|E|} \left(|W| + G(Q) \sum_{\chi \in E^\vee} (-1)^{\chi(\kappa) + \epsilon + Q(a_\chi)} \right), \quad (8.3)$$

where $G(Q) = \sum_{x \in W} (-1)^{Q(x)}$.

Proof. Expand both conditions with additive characters. Surjectivity of L kills the nontrivial characters in the nonquadratic sum. In the quadratic sum, complete the square:

$$Q(x) + b_Q(a_\chi, x) = Q(x + a_\chi) + Q(a_\chi).$$

This gives (8.3). $\square$

We now record the two pieces that replace an unrestricted count on all of $H^1(V)$.

The preimage of T in a scalar double cover need not have a normal complement to the new central involution. The cohomology class measuring that failure, and equivalently the variation of the scalar obstruction under T -twists, is called the *radical edge*.

Lemma 8.6 (Radical edge, variation formula, and descent). *Let $0 \rightarrow T \rightarrow M \rightarrow V \rightarrow 0$ be the simple-head sequence, and let*

$$p : \tilde{B} \twoheadrightarrow B$$

be a central double cover whose restriction to M has quadratic form with polar radical T and whose restriction to T is zero. The cover determines a canonical edge class

$$[\varepsilon] \in H^1(B/T, T^\vee).$$

Its restriction to V is zero, so it is inflated from a unique class

$$[\bar{\varepsilon}] \in H^1(C, T^\vee).$$

For every lower exact-image epimorphism $\rho : \Gamma \twoheadrightarrow C$ and every unrestricted M -lift f , twisting by $u \in Z_{\Gamma, \rho}^1(T)$ changes the scalar obstruction by

$$\text{ob}(f_u) = \text{ob}(f) + \langle [u], \rho^*[\bar{\varepsilon}] \rangle_\Gamma. \quad (8.4)$$

If $[\bar{\varepsilon}] \neq 0$, the free $Z_{\Gamma, \rho}^1(T)$ -action partitions the unrestricted M -lifts into orbits in each of which exactly one half satisfy the central relation. If $[\bar{\varepsilon}] = 0$, the cover descends to a central double cover of B/T ; conversely, descent forces the edge class to vanish.

Proof. We construct the edge cocycle, descend it from B/T to C , derive the exact variation formula, and then analyze the nonzero and zero edge cases.

Let z generate $\ker p$. Since both the square and commutator forms vanish on T , the preimage $\tilde{T} = p^{-1}(T)$ is elementary abelian. Choose a linear complement $s(T)$ to $\langle z \rangle$ in $\tilde{T}$.

For $\bar{b} \in B/T$, choose a lift $\hat{b} \in \tilde{B}$ and define $\varepsilon(\bar{b}) \in T^\vee$ by

$$\hat{b}s(t)\hat{b}^{-1} = s(\bar{b}t)z^{\varepsilon(\bar{b})(\bar{b}t)}. \quad (8.5)$$

The definition is independent of the chosen lift $\hat{b}$. Applying (8.5) successively to two elements shows

$$\varepsilon(\bar{b}_1\bar{b}_2) = \varepsilon(\bar{b}_1) + \bar{b}_1\varepsilon(\bar{b}_2),$$

so ε is a 1-cocycle. Replacing $s(t)$ by $s(t)z^{\ell(t)}$ changes it by the coboundary $d\ell$; hence its class is canonical.

If $\bar{b} \in V = M/T$, the exponent in (8.5) is the commutator pairing between a lift of $\bar{b}$ and t . This is the polar form of the restricted quadratic form, and it is zero because T is the polar radical. Hence $\text{res}_V[\varepsilon] = 0$. The group $V = M/T$ acts trivially on T , since M is elementary abelian. Inflation–restriction for $1 \rightarrow V \rightarrow B/T \rightarrow C \rightarrow 1$ therefore gives a unique $[\bar{\varepsilon}] \in H^1(C, T^\vee)$ inflating to $[\varepsilon]$. Choose a normalized cocycle representative $\bar{\varepsilon}$ of this class. The difference between ε and its inflation is a coboundary $d\ell$ for some $\ell \in T^\vee$; changing the complement $s(T)$ by ℓ therefore lets us assume from now on that $\varepsilon = \text{inf}(\bar{\varepsilon})$ as cocycles.

Fix a lower exact-image epimorphism $\rho : \Gamma \twoheadrightarrow C$, an M -lift $f : \Gamma \rightarrow B$, and lifts $\hat{f}(g) \in \tilde{B}$. For $u \in Z_{\Gamma, \rho}^1(T)$ put

$$f_u(g) = u(g)f(g), \quad \hat{f}_u(g) = s(u(g))\hat{f}(g).$$

Let ω_f and ω_{f_u} be the scalar central defects. Using (8.5), the cocycle identity $u(gh) = u(g) + \rho(g)u(h)$, and the fact that $s(T)$ is elementary abelian gives the exact identity

$$\omega_{f_u}(g, h) = \omega_f(g, h) + \bar{\varepsilon}(\rho(g))(\rho(g)u(h)). \quad (8.6)$$

For the local source, applying the invariant map identifies the second term with the ordinary cup pairing. For the candidate source, its evaluation on the two relation defects is the adjoint pairing supplied by the direct finite-word Fox–Heisenberg chain map of [Proposition 5.10](#). Thus (8.6) yields (8.4) in either source theory.

If $[\bar{\varepsilon}] \neq 0$, then $\rho^*[\bar{\varepsilon}] \neq 0$. Indeed, a pulled-back coboundary gives the same coboundary formula on every element of C because ρ is surjective. Perfect degree-one duality therefore makes

$$[u] \mapsto \langle [u], \rho^*[\bar{\varepsilon}] \rangle_\Gamma$$

a nonzero linear functional on $H_{\Gamma,\rho}^1(T)$, and hence on $Z_{\Gamma,\rho}^1(T)$. This group acts freely on each fibre of the map from M -lifts to their quotient modulo T . Equation (8.4) is one affine equation with nonzero linear part on every such orbit, so exactly half of its elements satisfy the scalar central relation.

Finally, if $[\bar{\varepsilon}] = 0$, then $[\varepsilon] = 0$. Alter the complement $s(T)$ so that the cocycle in (8.5) is identically zero. Then $s(T) \triangleleft \tilde{B}$, and

$$\tilde{B}/s(T) \twoheadrightarrow B/T$$

is a central double cover. Conversely, a descended cover supplies a normal complement to the preimage of T , so the edge class vanishes. $\square$

Assume henceforth that the edge vanishes. By [Lemma 6.21](#), the quotient B/T is identified with $V \rtimes C$. For the scalar pushout class under consideration, choose the base determinant class κ_q^0 attached to the descended nonsingular form on V . After a normalized change of section, write the descended class by the following equality of normalized 2-cocycles:

$$\kappa = \kappa_q^0 + \Gamma_{\gamma_\kappa} + \inf_C^{V \rtimes C} \delta_\kappa, \quad \gamma_\kappa \in Z^1(C, V^\vee), \quad \delta_\kappa \in Z^2(C, \mathbb{F}_2). \quad (8.7)$$

The original group B remains a possibly nonsplit extension of $V \rtimes C$ by T .

Lemma 8.7 (Affine T -lifting equation). *Choose the splitting $B/T \cong V \rtimes C$ above and let*

$$e \in H^2(C, T)$$

be the class obtained by pulling $B \rightarrow V \rtimes C$ back along the zero section $C \rightarrow V \rtimes C$. For a lower exact-image map $\rho : \Gamma \twoheadrightarrow C$, a class

$$c \in H_{\Gamma,\rho}^1(V)$$

is the V -coordinate of an actual M -lift if and only if

$$\partial_{\Gamma,\rho}(c) = \rho^*e \quad \text{in } H_{\Gamma,\rho}^2(T). \quad (8.8)$$

For every such class, the number of raw lifts above it is

$$\mu = |B_{\Gamma,\rho}^1(V)| |Z_{\Gamma,\rho}^1(T)|, \quad (8.9)$$

which is independent of ρ and has the same value on the two sources.

Proof. Represent the T -extension of $V \rtimes C$ by a normalized cocycle. Its pure $V \times V$ component is zero because M is elementary abelian; its mixed component is the module extension $0 \rightarrow T \rightarrow M \rightarrow V \rightarrow 0$; and its restriction to the zero section is e . Pulling this cocycle back along the graph of a V -cocycle gives exactly $\partial c + \rho^*e$, proving (8.8) in characteristic two.

Each cohomology class has $|B^1(V)|$ cocycle representatives, and each solvable T -defect equation has $|Z^1(T)|$ solutions. For the nontrivial simple V , $|B^1(V)| = 2^{\dim V}$; [Proposition 5.15](#), [Proposition 5.16](#) gives the common value of $|Z^1(T)|$. $\square$

Put

$$\mathcal{X}_T = (T^\vee)^C \cong H_{\Gamma, \rho}^2(T)^\vee.$$

For $\chi \in \mathcal{X}_T$, let

$$\gamma_\chi = \partial_C^\vee(\chi) \in H^1(C, V^\vee), \quad a_{\chi, \kappa} = (b_q^\flat)^{-1}(\gamma_\chi + \gamma_\kappa) \in H^1(C, V).$$

Let

$$\Theta_q^0(a_{\chi, \kappa}) = (a_{\chi, \kappa}, \text{id}_C)^* \kappa_q^0 \in H^2(C, \mathbb{F}_2) \quad (8.10)$$

be the base graph class, and define the single total target-side phase

$$\Delta_{\chi, \kappa} = \chi_* e + \delta_\kappa + \Theta_q^0(a_{\chi, \kappa}) \in H^2(C, \mathbb{F}_2). \quad (8.11)$$

Proposition 8.8 (Completed-square phase identity). *For either source, every lower exact-image map ρ , every $b \in H_{\Gamma, \rho}^1(V)$, and every $\chi \in \mathcal{X}_T$,*

$$Q_{\kappa, \Gamma, \rho}(b) + \langle \partial_{\Gamma, \rho} b + \rho^* e, \chi \rangle = Q_{\Gamma, \rho}^0(b + \rho^* a_{\chi, \kappa}) + \iota_\Gamma(\rho^* \Delta_{\chi, \kappa}). \quad (8.12)$$

Here Q^0 is the base determinant form and Q_κ is the obstruction for the actual scalar pushout class (8.7).

Proof. By Corollary 5.17,

$$\langle \partial_{\Gamma, \rho} b, \chi \rangle = \langle b, \rho^* \gamma_\chi \rangle_\Gamma.$$

Using (6.7), the left side of (8.12) is

$$Q_{\Gamma, \rho}^0(b) + \langle b, \rho^*(\gamma_\kappa + \gamma_\chi) \rangle_\Gamma + \iota_\Gamma(\rho^* \delta_\kappa) + \iota_\Gamma(\rho^*(\chi_* e)).$$

By the definition of $a_{\chi, \kappa}$, the middle linear term is the polar pairing $b_q(b, \rho^* a_{\chi, \kappa})$. Completing the square for the base quadratic form gives

$$Q^0(b) + b_q(b, \rho^* a_{\chi, \kappa}) = Q^0(b + \rho^* a_{\chi, \kappa}) + \iota_\Gamma(\rho^* \Theta_q^0(a_{\chi, \kappa})).$$

Adding the two inflated scalar terms proves (8.12). The class $\Delta_{\chi, \kappa}$ is defined on the finite target C ; no comparison map between the two source H^1 -spaces is involved. $\square$

8.1 The exact-image recursion

Fix a boundary-framed target $\mathcal{Y} = (Y, L_Y, \pi_Y, \theta_Y)$ whose marked kernel has a non-scalar simple head, and retain the notation (7.1). Abbreviate $e_\Gamma(Y) = e_\Gamma^\beta(\mathcal{Y})$, with the same convention for inherited exact-image subgroups and central covers. The successive finite operations are

$$\begin{aligned} C &\rightsquigarrow M\text{-lifts} \rightsquigarrow \text{proper-image subtraction} \\ &\rightsquigarrow \text{scalar tests of the } R\text{-obstruction} \rightsquigarrow \text{phase covers} \rightsquigarrow Y. \end{aligned}$$

The elementary stage counts all M -lifts over C and removes their proper images. The remaining R -valued obstruction is separated by invariant scalar pushouts; in the zero-edge case, completing the resulting quadratic sums replaces their scalar phases by central covers before the final Fourier inversion.

Lemma 8.9 (Elementary lifting and exact-image subtraction). *For every $\rho \in X_\Gamma(C)$,*

$$H_{\Gamma,\rho}^2(M) = 0, \quad |Z_{\Gamma,\rho}^1(M)| = 2^{2 \dim M}.$$

Consequently

$$e_\Gamma(B) = 2^{2 \dim M} e_\Gamma(C) - \sum_{\substack{J < B \\ J \rightarrow C}} e_\Gamma(J), \quad (8.13)$$

where each J has the boundary-framed structure induced from B . For every proper $J < B$ occurring in this sum,

$$|J \cap L_B| = |J \cap M| |L_C| \leq \frac{|M|}{2} |L_C| = \frac{|L_B|}{2} < |L_Y|. \quad (8.14)$$

Proof. The unique simple head of M is nontrivial, so $(M^\vee)^C = 0$ by Lemma 7.1. Candidate deformation duality and local Tate duality therefore give the displayed vanishing and lift multiplicity. Every relation-compatible M -lift has an exact image $J \leq B$ mapping onto C ; partitioning by that exact image gives (8.13). If $J < B$, then $J \cap M$ is a proper subgroup of the elementary abelian group M , and the exact sequence

$$1 \longrightarrow J \cap M \longrightarrow J \cap L_B \longrightarrow L_C \longrightarrow 1$$

gives (8.14). □

Lemma 8.10 (Scalar pushouts separate the complete obstruction). *Put*

$$\mathcal{X}_R = (R^\vee)^B = (R^\vee)^C.$$

For either source, $\mathcal{X}_R$ is canonically the dual of the R -valued obstruction space. If o is an R -valued lifting obstruction, then

$$o = 0 \iff \lambda_* o = 0 \text{ for every } \lambda \in \mathcal{X}_R.$$

For $0 \neq \lambda \in \mathcal{X}_R$, the corresponding scalar pushout is the central double cover

$$p_\lambda : B_\lambda = Y / \ker \lambda \twoheadrightarrow B.$$

Proof. By Lemma 7.2, R is elementary abelian and K centralizes R , so the Y -action on R factors through C . The two duality theorems give

$$H_{A,\rho}^2(R)^\vee \cong (R^\vee)^C, \quad H_{F,\rho}^2(R)^\vee \cong (R^\vee)^C.$$

For $\lambda \in (R^\vee)^C$, the subgroup $\ker \lambda$ is normal in Y and C acts trivially on $R / \ker \lambda$. Thus $Y / \ker \lambda \rightarrow Y / R$ is a central double cover, and its obstruction is the pairing with λ . Perfectness proves the equivalence. □

Separating the obstruction makes the final Frattini layer accessible to the same inversion.

Proposition 8.11 (Fourier inversion for the final Frattini obstruction). *Assume $R \neq 1$ and put*

$$z_R = |Z_{\Gamma,\rho}^1(R)| = 2^{2 \dim R + \dim \mathcal{X}_R}.$$

For $\lambda \in \mathcal{X}_R$, let $m_{\Gamma,\lambda}(B)$ be the number of exact-image maps to B for which the scalar pushout $\lambda_ o_R$ vanishes. Then $m_{\Gamma,0}(B) = e_\Gamma(B)$ and*

$$e_\Gamma(Y) = \frac{z_R}{|\mathcal{X}_R|} \sum_{\lambda \in \mathcal{X}_R} (2m_{\Gamma,\lambda}(B) - e_\Gamma(B)). \quad (8.15)$$

Proof. For a fixed exact-image map to B , the set of R -lifts, when nonempty, is a torsor of size z_R . Every such lift is surjective onto Y : if its image J maps onto Y/R , then $(J \cap K)R = K$, and $R = \Phi(K)$ forces $J \cap K = K$, hence $J = Y$. Fourier inversion (8.2) on the full R -valued obstruction, followed by the common lift multiplicity z_R , gives (8.15). $\square$

Lemma 8.12 (Proper-image subtraction for a scalar pushout). *Fix $0 \neq \lambda \in \mathcal{X}_R$. Let $Z_{\Gamma,\lambda}(B/C)$ count all compatible lifts of exact-image maps to C , without imposing generation in B . Then*

$$m_{\Gamma,\lambda}(B) = Z_{\Gamma,\lambda}(B/C) - \sum_{\substack{J \leq B \\ J \rightarrow C}} m_{\Gamma,\lambda}(J). \quad (8.16)$$

For every proper J in the sum,

$$8m_{\Gamma,\lambda}(J) = \sum_{\substack{\tilde{J} \leq p_{\lambda}^{-1}(J) \\ p_{\lambda}(\tilde{J})=J}} e_{\Gamma}(\tilde{J}). \quad (8.17)$$

Proof. Partition the relation-compatible maps to B by their exact image. This gives (8.16). For a fixed proper image J , an unobstructed map has exactly eight scalar lifts; partitioning those lifts by their exact images in $p_{\lambda}^{-1}(J)$ is Lemma 8.3 and gives (8.17). $\square$

The recursion now splits according to the radical edge of the character.

Proposition 8.13 (The nonzero radical-edge case). *If the radical edge of p_{λ} is nonzero, then*

$$Z_{\Gamma,\lambda}(B/C) = 2^{2 \dim M - 1} e_{\Gamma}(C). \quad (8.18)$$

Proof. By Lemma 8.6, the scalar obstruction varies by a nonzero affine functional on each free $Z_{\Gamma,\rho}^1(T)$ -orbit of unrestricted M -lifts. Exactly half of the $2^{2 \dim M}$ lifts over each $\rho \in X_{\Gamma}(C)$ satisfy the central relation, which is (8.18). $\square$

Proposition 8.14 (The zero radical-edge case). *Assume that the radical edge vanishes. Write the descended scalar pushout class as*

$$\kappa = \kappa_q^0 + \Gamma_{\gamma_{\kappa}} + \inf \delta_{\kappa} \quad \text{on } V \rtimes C,$$

and put

$$\mathcal{X}_T = (T^{\vee})^C, \quad r = \dim \mathcal{X}_T, \quad d = \dim V, \quad \mu = |B_{\Gamma,\rho}^1(V)| |Z_{\Gamma,\rho}^1(T)|.$$

For each lower map ρ ,

$$\begin{aligned} & \#\{c : \partial_{\Gamma,\rho} c = \rho^* e, \ Q_{\kappa,\Gamma,\rho}(c) = 0\} \\ &= \frac{1}{2^{r+1}} \left(2^d + G(Q^0) \sum_{\chi \in \mathcal{X}_T} (-1)^{\iota_{\Gamma}(\rho^* \Delta_{\chi,\kappa})} \right). \end{aligned} \quad (8.19)$$

Consequently

$$Z_{\Gamma,\lambda}(B/C) = \frac{\mu}{2^{r+1}} \left(2^d e_{\Gamma}(C) + G(Q^0) \sum_{\chi \in \mathcal{X}_T} s_{\Gamma}(\Delta_{\chi,\kappa}) \right), \quad (8.20)$$

where

$$s_{\Gamma}(\zeta) = \sum_{\rho \in X_{\Gamma}(C)} (-1)^{\iota_{\Gamma}(\rho^* \zeta)}.$$

Proof. The vanishing edge and Lemma 6.21 identify B/T with $V \rtimes C$. By Lemma 8.7, the admissible V -coordinates satisfy $\partial c = \rho^* e$ and each occurs with multiplicity μ . The map $\partial_{\Gamma,\rho}$ is surjective because $H_{\Gamma,\rho}^2(M) = 0$. Apply the constrained Gauss formula (8.3) and the completed-square identity (8.12); the common Gauss sum is supplied by Proposition 6.9, Proposition 6.18. This gives (8.19). Multiplication by μ and summation over $X_{\Gamma}(C)$ give (8.20). $\square$

A scalar class $\zeta \in H^2(C, \mathbb{F}_2)$ is represented by a central double cover of C . We call it the *phase cover* because its splitting indicator realizes the sign $(-1)^{\iota_{\Gamma}(\rho^* \zeta)}$ appearing in the Fourier sum.

Lemma 8.15 (Phase-cover transform). *For $\zeta \in H^2(C, \mathbb{F}_2)$, let $p_{\zeta} : C_{\zeta} \rightarrow C$ be the corresponding central double cover and let $n_{\Gamma,0}(\zeta)$ count the lower exact-image maps for which it is unobstructed. Then*

$$s_{\Gamma}(\zeta) = 2n_{\Gamma,0}(\zeta) - e_{\Gamma}(C) \quad (8.21)$$

and

$$8n_{\Gamma,0}(\zeta) = \sum_{\substack{\tilde{J} \leq C_{\zeta} \\ p_{\zeta}(\tilde{J}) = C}} e_{\Gamma}(\tilde{J}). \quad (8.22)$$

Proof. The first identity is the decomposition of the signed sum according to the two values of the scalar obstruction. The second is the exact eight-lift partition of Lemma 8.3, applied to p_{ζ} . $\square$

Lemma 8.16 (Strict decrease of recursive targets). *Every target occurring on the right sides of (8.13), (8.16), (8.17), and (8.22) has marked 2-kernel strictly smaller than $|L_Y|$. More precisely,*

$$|L_C| = \frac{|L_Y|}{|K|} < |L_Y|, \quad (8.23)$$

$$2|J \cap L_B| \leq |L_B| < |L_Y|$$

for a proper scalar-cover stratum, and

$$|L_{C_{\zeta}}| = 2|L_C| = \frac{2|L_Y|}{|K|} < |L_Y|. \quad (8.24)$$

The proper-image bound in the elementary stage is (8.14).

Proof. The first two inequalities follow from $R < K$ and from the fact that a proper subgroup of the elementary abelian group $M = K/R$ has index at least two. A scalar double cover of a proper $J < B$ has marked kernel $2|J \cap L_B|$, giving (8.23). Since the non-scalar simple head has dimension at least two, $|K| \geq 4$; hence a phase cover has marked kernel as in (8.24). $\square$

Strict decrease of the marked 2-kernel is what closes the recursion: every count reduces to boundary data and exact-image counts for smaller targets.

Theorem 8.17 (Closed exact-image recursion). *Suppose L_Y has a non-scalar simple head. The exact-image count $e_\Gamma(Y)$ is obtained from (8.13) and, when $R \neq 1$, from (8.15)–(8.22). Every exact-image count on the right-hand sides belongs to a boundary-framed target with strictly smaller marked 2-kernel. Hence the count is determined by the source interface of Corollary 6.19 and the exact-image counts for smaller targets.*

Proof. The elementary stage is Lemma 8.9. If $R = 1$, then $B = Y$. Otherwise Lemma 8.10, Proposition 8.11 reduce the final R -obstruction to its scalar pushouts. Proper images are removed by Lemma 8.12. The unrestricted scalar count is given by Proposition 8.13 when the radical edge is nonzero and by Proposition 8.14 when it vanishes. The signed scalar phases in the latter formula are exact-image counts by Lemma 8.15. Strict decrease is Lemma 8.16. $\square$

9 Proof of the boundary-framed theorem

The terminal fibre-product description and the closed recursion of Theorem 8.17 cover all finite boundary-framed targets. We prove Theorem 4.2 by strong induction on $|L_Y|$, simultaneously for all such targets; every recursive target carries the same map to $H \times E$.

Lemma 9.1 (Coprime-kernel subdirect products). *Let $A \twoheadrightarrow C$ and $B \twoheadrightarrow C$ be finite epimorphisms whose kernels have coprime orders. If a subgroup $J \leq A \times_C B$ projects onto both A and B , then $J = A \times_C B$.*

Proof. The kernel of $J \rightarrow A$ is a subgroup of $\ker(B \rightarrow C)$, and the kernel of $J \rightarrow B$ is a subgroup of $\ker(A \rightarrow C)$. Comparing the two order formulas for J shows that both kernels have their full possible orders, because the two ambient kernel orders are coprime. Thus J contains both kernels and is the full fibre product. $\square$

9.1 The terminal case: only trivial module factors

Lemma 9.2 (Targets with only trivial module factors). *Suppose every chief factor of L_Y is a trivial H -module. Let H_2 be the maximal 2-quotient of H and let $N = \ker(H \rightarrow H_2)$. Then N has odd order, has a unique normal lift $\tilde{N} \triangleleft Y$ centralizing L_Y , and*

$$Y \cong H \times_{H_2} Q, \quad Q = Y/\tilde{N} \text{ a finite 2-group.} \quad (9.1)$$

Projection to Q identifies exact-image subgroups of Y projecting onto H with subgroups $Q' \leq Q$ that map onto H_2 .

Proof. We construct and characterize the odd-order complement, identify Y as a fibre product, and then classify the exact-image subgroups.

The kernel N has odd order by Lemma 3.1. Let P be its inverse image in Y . Schur–Zassenhaus gives a complement $\tilde{N} \leq P$. Along a Y -chief series

$$1 = L_0 \triangleleft L_1 \triangleleft \cdots \triangleleft L_m = L_Y,$$

triviality of every factor gives $[L_i, \tilde{N}] \leq L_{i-1}$. Hence $[L_Y, {}_m\tilde{N}] = 1$. Coprime action gives

$$[L_Y, \tilde{N}] = [L_Y, \tilde{N}, \tilde{N}] = \cdots = [L_Y, {}_m\tilde{N}] = 1.$$

Thus $P = L_Y \times \tilde{N}$. Schur–Zassenhaus conjugacy says that all complements are L_Y -conjugate; since L_Y centralizes $\tilde{N}$, the complement is unique and therefore normal in Y .

The quotient $Q = Y/\tilde{N}$ is a 2-group. The two quotient maps define

$$Y \longrightarrow H \times_{H_2} Q.$$

Its kernel lies in both L_Y and $\tilde{N}$, hence is trivial, and the two groups have the same order. This proves (9.1).

If $J \leq Y$ projects onto H and $Q' = \text{pr}_Q(J)$, then $Q' \twoheadrightarrow H_2$. Inside $H \times_{H_2} Q'$, the subgroup J projects onto both factors. The two kernels have odd and 2-power orders, so Lemma 9.1 gives $J = H \times_{H_2} Q'$. Conversely, for every subgroup $Q' \leq Q$ mapping onto H_2 , the fibre product $H \times_{H_2} Q'$ is a subgroup of Y projecting onto H . $\square$

Let $q : Q \rightarrow H_2$ be the quotient map. The tame frame $\alpha : T_{\text{tame}} \rightarrow H$ induces, through the full unramified marking of Proposition 3.14, a map

$$\alpha_2 : \Pi \twoheadrightarrow H_2.$$

Every boundary decoration kills the odd group $\tilde{N}$ and factors through a map $\bar{\theta} : Q \rightarrow E$.

For either source, the universal property of the pullback gives a bijection between maps $f : \Gamma \rightarrow Y$ with fixed tame frame and maps $g : \Pi \rightarrow Q$ satisfying

$$qg = \alpha_2, \quad \bar{\theta}g = \bar{\psi}. \tag{9.2}$$

This uses the full $\mathbb{Z}_2$ marking: if $H_2 = C_{2^m}$, the first equation fixes the reduction of the unramified character modulo 2^m , not merely its parity.

The kernels of $H \rightarrow H_2$ and $Q \rightarrow H_2$ have coprime orders. Hence a subgroup of $H \times_{H_2} Q$ that projects onto both factors is the whole pullback. It follows that f is surjective exactly when g is surjective. More generally, projection to Q identifies every exact-image subgroup of Y projecting onto H with a subgroup of Q satisfying (9.2). Thus the terminal exact-image problems are genuinely identical for the two sources, by the fully marked isomorphism of their maximal pro-2 quotients.

9.2 The inductive step

The cohomological count in this step rests on the local Euler–Poincaré characteristic [1, Theorem (7.3.1)]; see also [15, Theorem I.2.8].

Assume that L_Y has a non-scalar simple head. The target tower and module sequence are those of (7.1). By Lemma 8.9, Lemma 8.16, the elementary lifting stage expresses $e_\Gamma(B)$ in terms of $e_\Gamma(C)$ and exact-image counts with smaller marked kernels. If $R = 1$, then $B = Y$ and the induction closes at this stage.

Suppose $R \neq 1$. The closed recursion of Theorem 8.17 expresses $e_\Gamma(Y)$ through the common lift multiplicities and Gauss signs of Corollary 6.19, together with exact-image counts for the smaller targets listed in Lemma 8.16. The induction hypothesis makes every such count equal for Γ_A and $G_{\mathbb{Q}_2}$. Thus

$$e_{\Gamma_A}^\beta(Y) = e_{G_{\mathbb{Q}_2}}^\beta(Y),$$

which completes the inductive step and proves Theorem 4.2.

10 Passage to all finite quotients

We remove the fixed boundary frame by partitioning the epimorphisms to a finite group according to their induced tame quotient map. Summing the fixed-frame equality gives equality of all finite epimorphism counts, and profinite reconstruction then proves the presentation theorem. The fixed-frame equality is now summed over the possible tame quotient maps of an arbitrary finite group. This gives equality of all finite epimorphism counts, to which the reconstruction lemma applies.

Lemma 10.1 (Exhaustion by tame boundary frames). *Let G be finite and put $L = O_2(G)$. For either source Γ , every epimorphism $f : \Gamma \twoheadrightarrow G$ determines a unique tame boundary frame*

$$T_{\text{tame}} \twoheadrightarrow G/L.$$

Conversely, with decoration $E = 0$, a boundary-framed epimorphism to G with a fixed tame frame is exactly an ordinary epimorphism to G inducing that frame. Distinct tame frames give disjoint sets of epimorphisms.

Proof. The characteristic marked wild subgroup maps into $O_2(G) = L$, so every epimorphism has a well-defined quotient map to G/L . By [Proposition 3.2](#), this quotient map is a tame frame. Conversely, when $E = 0$ there is no extra decoration, and the boundary-framed condition records only the induced tame quotient map and the marked 2-kernel. Thus a boundary-framed epimorphism with target G is precisely an ordinary epimorphism inducing the chosen frame. The induced quotient map $T_{\text{tame}} \rightarrow G/L$ is unique, so the fixed-frame sets are disjoint as the frame varies. $\square$

Let G be finite and put $L = O_2(G)$. By [Lemma 10.1](#), the ordinary surjection set is the disjoint union of the fixed-frame surjection sets over all tame frames $T_{\text{tame}} \rightarrow G/L$. Applying [Theorem 4.2](#) to each frame with $E = 0$, summing over the finite set of frames, and using [Proposition 2.3](#), gives

$$|\text{Sur}(\Gamma_A, G)| = |\text{Sur}(G_{\mathbb{Q}_2}, G)| \tag{10.1}$$

for every finite group G .

The candidate is topologically generated by four elements. The one-sided reconstruction lemma [Lemma 2.5](#) applied to [\(10.1\)](#) gives $\Gamma_A \cong G_{\mathbb{Q}_2}$, proving [Theorem 1.2](#).

11 Completion and independence from the computational verification

We record that every mathematical input to the presentation theorem has been proved, identify the precise role of the full unramified marking, and verify that the separate finite computation of the final word is not used in the proof. All mathematical inputs to the presentation theorem have now been proved. This section records where the full unramified marking enters and verifies that the separate finite computation mentioned in [Remark 1.3](#) is not used in the argument.

The marked local normalization is proved in [Sections 3.1–3.3](#); no separate local insertion remains. Its proof uses Labute’s abstract classification, local reciprocity, the intrinsic Demushkin orientation, and the peripheral cyclotomic lifting of [Lemma 3.7](#), [Lemma 3.8](#), [Proposition 3.9](#). Consequently the maximal pro-2 quotient of the candidate and $G_{\mathbb{Q}_2}(2)$ agree as marked pairs over the full quotient $\mathbb{Z}_2$.

The finite-word duality, local determinant calculation, constrained Fourier–Gauss transform, strict exact-image induction, and profinite reconstruction are independent of the separate computational

verification of the final word. The full integral marking is used only in the construction of the common boundary (4.1) and in the terminal pro-2 comparison, and those inputs are now supplied by Proposition 1.1, Corollary 3.12. Thus the mathematical proof of Theorem 1.2 does not depend on the unreproduced computation mentioned in Remark 1.3.

A natural continuation of the formalization is to drive the axiom census to zero and to upstream the reusable theory. Each remaining axiom—local Euler characteristic, the dyadic Hilbert symbol, duality inputs, the Demushkin classification [2]—is a well-scoped target, so the paper induces a prioritized worklist for formalizing local class field theory at $p = 2$.

12 Further questions

The presentation, the counting route behind it, and the formalization each suggest continuations. We record the questions we consider most tractable or most consequential; a formalization-specific continuation is stated at the end of Section 11.

1. *Uniformity across dyadic fields.* Extend Theorem 1.2 to every finite extension $K/\mathbb{Q}_2$ without rebuilding the proof field by field. The boundary formalism and counting induction are already uniform, while the marked Demushkin group [2], its cyclotomic orientation, and the quadratic comparison vary through the standard local data $[K : \mathbb{Q}_2]$, the image of the cyclotomic character, and the filtered square-class group with its Hilbert pairing. The central question is whether the candidate relator and the quadratic Gauss-sign calculation can be formulated once over this package of invariants. A positive answer would give a uniform theorem for all dyadic fields; a negative answer should identify the additional invariant and reduce the problem to finitely many structural types, rather than one calculation per field. The seven quadratic extensions of $\mathbb{Q}_2$, which lie outside Diekert’s hypotheses [14], are a useful stress test for such a uniform formulation; Schneider’s partial structural results [19] provide a second comparison point.
2. *Constraints on Jannsen–Wingberg-type presentations.* Classify the allowable wild relator in a dyadic analogue of the Jannsen–Wingberg presentation. Here *Jannsen–Wingberg shape* means the structural ansatz of [1, Theorem 7.5.14]: tame generators σ, τ with $\sigma\tau\sigma^{-1} = \tau^q$; wild generators whose closed normal subgroup is pro- p ; and one further relation expressing x_0^q as a cyclotomic norm word in x_0, τ , a p -power, and commutators whose partners are constrained by the tame cyclotomic action. Determine, modulo boundary-preserving changes of wild generators, which 2-adic analogues of this ansatz can present $G_{\mathbb{Q}_2}$. Low central quotients, the evaluated Fox Jacobian, and the quadratic Gauss sums should give necessary constraints on the relator. Constraints of this kind would also give criteria for evaluating the independently proposed candidate presentations mentioned in the introduction.
3. *Deformation rings from the presentation.* Compute universal deformation rings of residual representations $\bar{\rho} : G_{\mathbb{Q}_2} \rightarrow \mathrm{GL}_d(k)$ directly from the presentation. Substitute universal framed matrices into the two relator words and complete at $\bar{\rho}$; the evaluated Fox Jacobian is the linearization of these equations, matching Mazur’s description of the tangent space by $H^1(G_{\mathbb{Q}_2}, \mathrm{ad} \bar{\rho})$ and of the relation ideal by an obstruction map into H^2 [21, Section 1.6, Proposition 2], while higher Fox terms retain the full power-series equations. Compare the resulting completed rings, and their quotients imposing Hodge conditions, with the formal completions of the Emerton–Gee stack and its crystalline or semistable substacks [22, Section

1.1 and Theorem 1.2.4]. This would turn the local geometry of their moduli spaces at $G_{\mathbb{Q}_2}$ -representations into explicit word calculus: first recover known two-dimensional crystalline deformation rings as a consistency check, then compute new $p = 2$ examples.

4. *Explicit counting formulas.* Extract from the Fourier-inversion stage an explicit algorithm—ideally a closed form—for $|\text{Sur}(G_{\mathbb{Q}_2}, G)|$ for an arbitrary finite group G , refined by inertia and higher ramification data. The quotient $|\text{Sur}(G_{\mathbb{Q}_2}, G)|/|\text{Aut}(G)|$ counts G -Galois extensions, whereas Monge’s formula [24, Theorem 1] counts isomorphism classes of extensions with prescribed ramification index e and inertia degree f , equivalently conjugacy classes of the corresponding open subgroups. Find an explicit passage between these counts using subgroup and normalizer data, and reconcile it with the Krasner–Serre mass-formula tradition [23]; then refine the comparison by discriminant and upper ramification breaks. Tables for small G , cross-linked to the LMFDB’s 2-adic fields data and the machine-readable presentation, would make both kinds of enumeration computable from the same source.
5. *Ramification filtration and canonical wild generators.* Give the wild generators x_0, x_1 a canonical Galois-theoretic characterization up to conjugacy, and locate the ramification filtration in upper numbering word-theoretically in the presentation. The presentation retains only the tame–wild dichotomy; recovering the filtration would connect it to the local anabelian results [25], in which the filtration is the essential extra datum.
6. *The outer automorphism group.* Compute $\text{Out}(G_{\mathbb{Q}_2})$ —or at least its action on the marked tame–pro-2 data—directly from the presentation. Automorphism groups of 2-adic local Galois groups are studied in [16]; a finite presentation plus the Fox toolkit makes this a concrete word problem.
7. *The counting route as a general technique.* Establish the counting route as a general profinite technique. The pipeline—a candidate assembled from admissible finite quotients, the one-sided reconstruction lemma, a lifting-theory comparison, and Fourier induction—uses nothing specific to $\mathbb{Q}_2$ before the quadratic stage. Test cases: a machine-checkable re-proof of the odd- p theorem [12], and maximal pro- p Galois groups with restricted ramification, where Koch-type presentations supply candidates. Separately: is there an a priori bound on the induction, turning the count comparison into a finite decision procedure?

Normalized cochains, Shapiro maps, corestriction, and the Evens normThe formulas in Lemma 6.13, Lemma 6.15 depend on choices of left action, coset representatives, and normalized sections. This appendix fixes those conventions at cochain level and gives the explicit normalized boundaries produced by every change of section or transversal.

The general bar and transfer constructions are standard; compare [5, Chapter III and Chapter VII] and [9].

.1 Normalized inhomogeneous cochains

All coefficient modules are left modules. If G is profinite and M is a finite discrete continuous left G -module, then $C^n(G, M)$ denotes the group of continuous normalized inhomogeneous n -cochains. Thus a cochain of positive degree vanishes whenever one of its arguments is 1, and $C^0(G, M) = M$.

In additive notation the differential is

$$(df)(g_1, \dots, g_{n+1}) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \dots, g_n), \quad (.1)$$

and, in degree zero,

$$(dm)(g) = gm - m. \quad (.2)$$

All coefficient groups in the normalization-sensitive calculations of [Lemma 6.13](#), [Lemma 6.15](#) are $\mathbb{F}_2$ -vector spaces, so the minus signs in (.1) disappear there.

Let $\mu : A \times B \rightarrow C$ be a G -equivariant bilinear pairing of left G -modules. Our cup-product convention is

$$(f \smile g)(g_1, \dots, g_{p+q}) = \mu(f(g_1, \dots, g_p), (g_1 \cdots g_p)g(g_{p+1}, \dots, g_{p+q})). \quad (.3)$$

In particular, for trivial scalar coefficients, $(\alpha \smile \beta)(n_1, n_2) = \alpha(n_1)\beta(n_2)$.

We use the following three levels of assertion.

1. “Equal as normalized cochains” means equality in $C^n(G, M)$.
2. “Equal up to the displayed normalized coboundary” means $f - g = dh$ for the particular normalized cochain h that is written down.
3. “Equal in cohomology” means only $[f] = [g]$ in $H^n(G, M)$.

No cochain-level assertion in the main text is intended to pass silently from one of these levels to another.

.2 Conjugation and semidirect products

The group-theoretic exponent convention remains

$$x^g = g^{-1}xg.$$

It is distinct from the notation gm for the given left action of G on a module M . We write the semidirect product as $M \rtimes G$, with elements (m, g) and multiplication

$$(m, g)(n, h) = (m + gn, gh), \quad (m, g)^{-1} = (-g^{-1}m, g^{-1}). \quad (.4)$$

Consequently

$$(0, g)(m, 1)(0, g)^{-1} = (gm, 1), \quad (m, 1)^{(0, g)} = (g^{-1}m, 1).$$

Thus the manuscript’s right-conjugation notation acts on the normal module by g^{-1} , whereas the semidirect-product law uses the specified left action by g .

Let $\rho : \Gamma \rightarrow G$ and $b : \Gamma \rightarrow M$. The graph map

$$\Phi_{b, \rho} : \Gamma \longrightarrow M \rtimes G, \quad \gamma \longmapsto (b(\gamma), \rho(\gamma)) \quad (.5)$$

is a homomorphism if and only if

$$b(\gamma\eta) = b(\gamma) + \rho(\gamma)b(\eta). \quad (.6)$$

This is the graph-pullback convention used in [\(6.4\)](#), [\(6.40\)](#), and [\(6.51\)](#).

.3 The left-regular coordinate convention

For a finite group H we identify $\mathbb{F}_2[H]$ with the functions $x = (x_h)_{h \in H}$ and use the left-regular action

$$(cx)_h = x_{c^{-1}h}. \quad (.7)$$

Hence

$$(\bar{\gamma}x)_h = x_{\bar{\gamma}^{-1}h}.$$

This is the source of the index $\bar{\gamma}^{-1}h$ in the Shapiro cocycle and in the cup/corestriction formula. Right multiplication of the coordinate index is a different operator:

$$(R_g x)_h = x_{hg}. \quad (.8)$$

The operators R_g commute with the left-regular H -action. The monomial $x_u x_{ug}$ is therefore naturally indexed by cosets written $u\langle g \rangle$, which explains the right multiplication by g and the coset convention in [Lemma 6.2](#).

.4 The normalized Shapiro cochain map

Let

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1$$

be exact with H finite. Choose a normalized section $h \mapsto \tilde{h}$, with $\tilde{1} = 1$, and write $\bar{\gamma} = \pi(\gamma)$. Define the transition element

$$\lambda_h(\gamma) = \tilde{h}^{-1} \gamma \widetilde{\bar{\gamma}^{-1}h} \in N. \quad (.9)$$

It is normalized and satisfies

$$\lambda_h(\gamma\eta) = \lambda_h(\gamma) \lambda_{\bar{\gamma}^{-1}h}(\eta). \quad (.10)$$

For a normalized scalar cochain $f \in C^n(N, \mathbb{F}_2)$ put

$$(\text{Sh}_H^n f)(\gamma_1, \dots, \gamma_n)_h = f(\lambda_{h_0}(\gamma_1), \dots, \lambda_{h_{n-1}}(\gamma_n)), \quad h_i = \bar{\gamma}_1 \cdots \bar{\gamma}_i^{-1} h. \quad (.11)$$

Equation [\(.10\)](#) gives $d \text{Sh}_{\tilde{H}} = \text{Sh}_{\tilde{H}} d$ directly. Thus [\(.11\)](#) fixes a full normalized cochain map, although the main text uses it only in degree one, together with the degree-two corestriction formula below.

For $\alpha \in Z^1(N, \mathbb{F}_2)$ this specializes to

$$\text{Sh}_{\tilde{H}}(\alpha)(\gamma)_h = \alpha\left(\tilde{h}^{-1} \gamma \widetilde{\bar{\gamma}^{-1}h}\right). \quad (.12)$$

Indeed, by [\(.10\)](#),

$$\begin{aligned} \text{Sh}(\alpha)(\gamma\eta)_h &= \text{Sh}(\alpha)(\gamma)_h + \text{Sh}(\alpha)(\eta)_{\bar{\gamma}^{-1}h} \\ &= \text{Sh}(\alpha)(\gamma)_h + (\bar{\gamma} \text{Sh}(\alpha)(\eta))_h, \end{aligned}$$

which is exactly the left-module cocycle identity. Formula [\(6.39\)](#) is [\(.12\)](#) for $H = C_2$.

.5 The convention for $g\alpha$

Let $g \in H$, choose its section lift $\tilde{g}$, and let $\alpha \in Z^1(N, \mathbb{F}_2)$. We define

$$(g\alpha)(n) = \alpha(\tilde{g}^{-1}n\tilde{g}) \quad (n \in N). \quad (.13)$$

Because α has trivial scalar coefficients, this does not depend on replacing $\tilde{g}$ by another lift. It is the usual left H -action on $H^1(N, \mathbb{F}_2)$ expressed in the manuscript's conjugation convention.

For $h \in H$ put

$$a_h^{(g)} = \tilde{h}g^{-1}\tilde{h}\tilde{g} \in N, \quad q_{\alpha,g}(h) = \alpha(a_h^{(g)}). \quad (.14)$$

If $k = \tilde{\gamma}^{-1}h$, then

$$\lambda_{hg}(\gamma) = a_h^{(g)}\tilde{g}^{-1}\lambda_h(\gamma)\tilde{g}(a_k^{(g)})^{-1}.$$

Applying α gives the exact normalized-cochain identity

$$\text{Sh}(g\alpha) = R_g \text{Sh}(\alpha) + dq_{\alpha,g}, \quad (.15)$$

where $(dq_{\alpha,g})(\gamma)_h = q_{\alpha,g}(h) + q_{\alpha,g}(\tilde{\gamma}^{-1}h)$. If the section is right- g -equivariant, namely $\tilde{hg} = \tilde{h}\tilde{g}$ for all relevant h , then $q_{\alpha,g} = 0$ and (.15) is strict:

$$\text{Sh}(g\alpha)(\gamma)_h = \text{Sh}(\alpha)(\gamma)_{hg}.$$

No such equivariance is assumed silently when it is unavailable.

For later use, let $A = \text{Sh}(\alpha)$ and $B = \text{Sh}(\beta)$. With the H -invariant trace pairing $\langle x, y \rangle = \sum_h x_h y_h$, the free-orbit cochain formed with the right-shifted coordinate $R_g B$ differs from the cochain formed with $\text{Sh}(g\beta)$ by the displayed boundary

$$\text{tr}(A \smile R_g B) - \text{tr}(A \smile \text{Sh}(g\beta)) = d\mathfrak{b}_{\alpha,\beta,g}, \quad (.16)$$

where

$$\mathfrak{b}_{\alpha,\beta,g}(\gamma) = \sum_{h \in H} A(\gamma)_h q_{\beta,g}(\tilde{\gamma}^{-1}h). \quad (.17)$$

This follows from (.3), (.15), and the Leibniz rule. It is the precise cochain correction behind (6.46).

.6 Corestriction in degrees one and two

Let $K \leq G$ have finite index and let $\Omega = G/K$, with cosets written $t_\omega K$. Choose a normalized transversal $\{t_\omega\}_{\omega \in \Omega}$ and define

$$\ell_\omega(\gamma) = t_\omega^{-1}\gamma t_{\gamma^{-1}\omega} \in K. \quad (.18)$$

Then

$$\ell_\omega(\gamma\eta) = \ell_\omega(\gamma)\ell_{\gamma^{-1}\omega}(\eta). \quad (.19)$$

For scalar normalized cochains, the bar-corestriction map is

$$(\text{cor}_T^n f)(\gamma_1, \dots, \gamma_n) = \sum_{\omega \in \Omega} f(\ell_{\omega_0}(\gamma_1), \dots, \ell_{\omega_{n-1}}(\gamma_n)), \quad \omega_i = (\gamma_1 \cdots \gamma_i)^{-1}\omega. \quad (.20)$$

In degrees one and two this reads

$$(\text{cor}_{\mathcal{T}} \alpha)(\gamma) = \sum_{\omega \in \Omega} \alpha(\ell_{\omega}(\gamma)), \quad (.21)$$

$$(\text{cor}_{\mathcal{T}} \nu)(\gamma, \eta) = \sum_{\omega \in \Omega} \nu(\ell_{\omega}(\gamma), \ell_{\gamma^{-1}\omega}(\eta)). \quad (.22)$$

Equation (.19) gives

$$d \text{cor}_{\mathcal{T}} = \text{cor}_{\mathcal{T}} d. \quad (.23)$$

Equivalently, when $K = N \triangleleft G$ and $H = G/N$, corestriction is the coordinate-sum map after the Shapiro map.

In particular, for $\alpha, \beta \in Z^1(N, \mathbb{F}_2)$,

$$\text{cor}_N^G(\alpha \smile \beta)(\gamma, \eta) = \sum_{h \in H} \text{Sh}(\alpha)(\gamma)_h \text{Sh}(\beta)(\eta)_{\bar{\gamma}^{-1}h}. \quad (.24)$$

This is the exact normalized-cochain identity used in (.48).

For (.50), take $G = G_F$, $K = G_{K_0}$, $\Omega = H/J$, and representatives written uJ . The relation $\bar{\gamma}^{-1}u = \pi_{\bar{\gamma}}(u)g^{\epsilon_{\bar{\gamma}}(u)}$ means that $\gamma^{-1}(uJ) = \pi_{\bar{\gamma}}(u)J$, so (.22) becomes exactly (.50).

.7 Changing sections and transversals

We first record the degree-one Shapiro correction. Let $\tilde{h}' = \tilde{h}a_h$ with $a_h \in N$ and $a_1 = 1$. Then

$$\lambda'_h(\gamma) = a_h^{-1} \lambda_h(\gamma) a_{\bar{\gamma}^{-1}h}.$$

For $\alpha \in Z^1(N, \mathbb{F}_2)$ define the normalized 0-cochain $q_{\alpha} \in \mathbb{F}_2[H]$ by $q_{\alpha}(h) = \alpha(a_h)$. Then

$$\text{Sh}_{\tilde{H}'}(\alpha) - \text{Sh}_{\tilde{H}}(\alpha) = dq_{\alpha} \quad \text{in } C^1(G, \mathbb{F}_2[H]). \quad (.25)$$

Thus the change of Shapiro lift is not left as an unspecified cohomology ambiguity.

For all degrees there is an explicit prism homotopy. Let $K \leq G$ be as in Subsection .6, and let $t'_{\omega} = t_{\omega}a_{\omega}$, $a_{\omega} \in K$, be a second normalized transversal indexed by the same coset set Ω . For a normalized n -cochain f on K , write

$$f^h(y_0, \dots, y_n) = f(y_0^{-1}y_1, \dots, y_{n-1}^{-1}y_n)$$

whenever all consecutive ratios lie in K . Given $\gamma_1, \dots, \gamma_{n-1} \in G$, put

$$p_i = \gamma_1 \cdots \gamma_i, \quad \omega_i = p_i^{-1}\omega, \quad y_i = p_i t_{\omega_i}, \quad y'_i = p_i t'_{\omega_i} = y_i a_{\omega_i}.$$

Define

$$(\mathfrak{c}_{\mathcal{T}, \mathcal{T}'}^n f)(\gamma_1, \dots, \gamma_{n-1}) = \sum_{\omega \in \Omega} \sum_{i=0}^{n-1} f^h(y_0, \dots, y_i, y'_i, \dots, y'_{n-1}). \quad (.26)$$

Over general coefficients the inner sum has the usual signs $(-1)^i$; over $\mathbb{F}_2$ they are all $+1$. The simplicial prism identity gives

$$\text{cor}_{\mathcal{T}'} f - \text{cor}_{\mathcal{T}} f = d\mathfrak{c}_{\mathcal{T}, \mathcal{T}'} f + \mathfrak{c}_{\mathcal{T}, \mathcal{T}'}(df). \quad (.27)$$

This is the promised cochain-level change-of-transversal formula.

For a normalized 2-cochain ν , the degree-one transition cochain in (.26) is especially concrete. Write $\ell_\omega, \ell'_\omega$ for the transition elements associated to the two transversals. Then

$$\begin{aligned} (\mathfrak{c}_{\mathcal{T}, \mathcal{T}'} \nu)(\gamma) &= \sum_{\omega \in \Omega} (\nu(a_\omega, \ell'_\omega(\gamma)) \\ &\quad + \nu(\ell_\omega(\gamma), a_{\gamma^{-1}\omega})). \end{aligned} \quad (.28)$$

It is normalized. If $d\nu = 0$, then

$$\text{cor}_{\mathcal{T}'} \nu - \text{cor}_{\mathcal{T}} \nu = d(\mathfrak{c}_{\mathcal{T}, \mathcal{T}'} \nu). \quad (.29)$$

The same prism without the outer sum over Ω gives the corresponding homotopy between the two Shapiro cochain maps; in degree one it is precisely (.25).

We now specialize these formulas to the two cocycles used in the manuscript. Let $J = \langle g \rangle$, $M = \mathbb{F}_2[H]$, and write $p : M \rightarrow \mathbb{F}_2[J]$ for the projection to the coordinates $1, g$. Put

$$\theta_g = (p \rtimes 1)^* \kappa_J \in Z^2(M \rtimes J, \mathbb{F}_2).$$

For a transversal $\mathcal{R}$ of the cosets uJ , represented in $M \rtimes H$ by $\widehat{u} = (0, u)$, direct substitution into (.22) gives the literal cochain equality

$$\kappa_g^{\mathcal{R}} = \text{cor}_{\mathcal{R}} \theta_g. \quad (.30)$$

Indeed, the transition element for $(x, c) \in M \rtimes H$ is

$$\widehat{u}^{-1}(x, c) \widehat{\pi_c(u)} = (u^{-1}x, g^{\epsilon_c(u)}),$$

and its two selected coordinates are x_u, x_{ug} ; the two terms of κ_J reproduce (.6.11) and (.6.12).

Let $\mathcal{R}'$ be a second transversal, indexed so that $u' = uj_u$ with $j_u \in J$, and put $A_u = (0, j_u) \in M \rtimes J$. Formula (.28) therefore gives the explicit normalized 1-cochain

$$\begin{aligned} \Lambda_{\mathcal{R}, \mathcal{R}'}^g(x, c) &= \sum_{u \in \mathcal{R}} (\theta_g(A_u, \ell'_u(x, c)) \\ &\quad + \theta_g(\ell_u(x, c), A_{\pi_c(u)})) \end{aligned} \quad (.31)$$

for which

$$\kappa_g^{\mathcal{R}'} - \kappa_g^{\mathcal{R}} = d\Lambda_{\mathcal{R}, \mathcal{R}'}^g. \quad (.32)$$

Thus the half-orbit change of transversal has a specified boundary, not only an abstract uniqueness argument.

Likewise, in the field-theoretic situation of Lemma 6.15, put

$$\theta_\alpha = (\text{Sh}_{K/K_0}(\alpha), \rho_0)^* \kappa_J \in Z^2(G_{K_0}, \mathbb{F}_2)$$

and define $\nu_{\alpha, \mathcal{R}} = \text{cor}_{\mathcal{R}} \theta_\alpha$. If the lifts of the representatives change from $\tilde{u}$ to $\tilde{u}' = \tilde{u}a_u$ with $a_u \in G_{K_0}$, then

$$\begin{aligned} \Lambda_{\mathcal{R}, \mathcal{R}'}^\alpha(\gamma) &= \sum_{u \in \mathcal{R}} (\theta_\alpha(a_u, \ell'_u(\gamma)) \\ &\quad + \theta_\alpha(\ell_u(\gamma), a_{\pi_{\tilde{\gamma}}(u)})) \end{aligned} \quad (.33)$$

satisfies

$$\nu_{\alpha, \mathcal{R}'} - \nu_{\alpha, \mathcal{R}} = d\Lambda_{\mathcal{R}, \mathcal{R}'}^\alpha. \quad (.34)$$

With the compatible nested transversals chosen in the proof of [Lemma 6.15](#), equation [\(6.51\)](#) is a literal equality of normalized cochains. Equations [\(.32\)](#) and [\(.34\)](#) give the correction for arbitrary choices.

Finally, consider the index-two formula itself. If the lift $\tilde{s}$ is replaced by $\tilde{s}' = \tilde{s}a$, $a \in N$, then [\(.25\)](#) gives $b' = b + dq$ with $q_1 = 0$ and $q_s = \alpha(a)$. Put $A = (q, 1) \in E \rtimes J$ and

$$\Phi_b(\gamma) = (b(\gamma), \bar{\gamma}), \quad \Phi_{b'}(\gamma) = (b'(\gamma), \bar{\gamma}).$$

Then

$$\Phi_{b'}(\gamma) = A^{-1}\Phi_b(\gamma)A.$$

For $p \in E \rtimes J$ define

$$r_A(p) = \kappa_J(A^{-1}, A) + \kappa_J(A^{-1}, p) + \kappa_J(A^{-1}p, A). \quad (.35)$$

This is the central coordinate of conjugation by the canonical lift of A in the extension represented by κ_J . Therefore

$$\nu_{\alpha, \tilde{s}'} - \nu_{\alpha, \tilde{s}} = d(r_A \circ \Phi_b). \quad (.36)$$

This displays the normalized boundary asserted after [\(6.40\)](#).

.8 The index-two specialization and the Evens normalization

Let $J = \langle s \rangle \cong C_2$ act on $E_2 = \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_s$ by swapping the two coordinates. Put

$$q_{\text{hyp}}(x_1, x_s) = x_1 x_s, \quad f_J(x, y) = x_1 y_s, \quad m_1 = 0, \quad m_s(y) = y_1 y_s,$$

and

$$\kappa_J((x, c), (y, d)) = f_J(x, cy) + m_c(y). \quad (.37)$$

The identities

$$\begin{aligned} m_s(v + w) + m_s(v) + m_s(w) &= f_J(sv, sw) + f_J(v, w), \\ m_s(sv) + m_s(v) &= 0 \end{aligned}$$

show directly, after expanding $d\kappa_J((x, c), (y, d), (z, e))$, that $d\kappa_J = 0$. The cochain is normalized. On the zero section it is zero, and on the fibre

$$\kappa_J((x, 1), (x, 1)) = f_J(x, x) = x_1 x_s,$$

so its square map is q_{hyp} . The lifts of e_1, e_s are involutions, their commutator is the central involution, and their product has nontrivial square; the fibre extension is therefore D_8 .

We normalize the homogeneous index-two Evens norm by requiring

$$N_N^G(1 + [\alpha]) = 1 + \text{cor}_N^G[\alpha] + N_N^{G, \text{Ev}}([\alpha]) \quad (.38)$$

for $[\alpha] \in H^1(N, \mathbb{F}_2)$. For the universal inclusion $E_2 \leq E_2 \rtimes J$, Mackey restriction gives

$$\text{res}_{E_2} N_{E_2}^{E_2 \rtimes J, \text{Ev}}(e_1^\vee) = e_1^\vee e_s^\vee, \quad \text{res}_J N_{E_2}^{E_2 \rtimes J, \text{Ev}}(e_1^\vee) = 0.$$

Since

$$H^1(J, E_2^\vee) = \ker(1 + s) / \operatorname{im}(1 + s) = 0,$$

the low-degree sequence for $1 \rightarrow E_2 \rightarrow E_2 \rtimes J \rightarrow J \rightarrow 1$ shows that these two restrictions determine the class. They are precisely the fibre and zero-section restrictions of (37); hence

$$[\kappa_J] = N_{E_2}^{E_2 \rtimes J, \operatorname{Ev}}(e_1^\vee). \quad (.39)$$

Now let $N \triangleleft G$ have index two and let $b = \operatorname{Sh}(\alpha)$ be the cocycle (12). By Subsection .2, the graph $\gamma \mapsto (b(\gamma), \bar{\gamma})$ is a homomorphism. Pulling back (37) gives, literally,

$$\nu_\alpha(\gamma, \eta) = b(\gamma)_1 b(\eta)_{\bar{\gamma}^{-1}s} + \varepsilon(\bar{\gamma}) b(\eta)_1 b(\eta)_s, \quad (.40)$$

where $\varepsilon(1) = 0$ and $\varepsilon(s) = 1$. Naturality of the Evens norm in this pullback square and (39) give

$$[\nu_\alpha] = N_N^{G, \operatorname{Ev}}([\alpha]).$$

For profinite Galois groups, every cochain in this construction factors through a finite quotient, and the displayed finite-group normalization is applied there.

The convention (38) is also the one used in the Evens–Kahn formula: for a quadratic extension L/k , the degree-two term in the relative total Stiefel–Whitney identity is

$$w(\operatorname{Tr}_{L/k}\langle a \rangle) = w(\operatorname{Tr}_{L/k}\langle 1 \rangle)(1 + \operatorname{cor}_{L/k}[a] + N_{L/k}^{\operatorname{Ev}}([a])).$$

Thus the normalization in this appendix is exactly the normalization used in (6.53); no additional sign or inverse convention enters Lemma 6.16.

.9 Convention audit for the orbit formulas

The following table records where each convention-sensitive formula in Lemma 6.13, Lemma 6.15 is fixed.

Manuscript formula	Convention-dependent input	Appendix result
(6.39)	normalized Shapiro section and the index $\bar{\gamma}^{-1}h$	Subsection .3, Subsection .5
(6.40)	semidirect graph pullback and the orientation of the two coordinates	Subsection .2, Subsection .5
(6.46)	the definition of $g\alpha$ and the right-coordinate shift	Subsection .5
(6.48)	cup product and normalized bar corestriction	Subsection .1, Subsection .5
(6.50)	the coset action and degree-two transition elements	Subsection .6, Subsection .5
(6.51)	compatibility of the two Shapiro stages and change of transversal	Subsection .5, Subsection .5

Finite Fox–Heisenberg rules and word ledger We collect the finite word-evaluation rules underlying the first- and second-order deformation calculations. Direct multiplication in the relevant semidirect, Heisenberg, and extraspecial groups yields the Jacobian, mixed pairing, and candidate quadratic-form ledgers, while the finite-exponent argument shows that these calculations do not depend on the chosen integer representative of the profinite exponent. This appendix starts from the finite semidirect, Heisenberg, and extraspecial groups used in the first- and second-order calculations. It records the literal word-evaluation identities that supply the Jacobian, mixed pairing, and candidate quadratic form in Section 5, Section 6.

The rules below are those used in [Lemma 5.4](#), [Lemma 5.5](#), [Lemma 5.14](#), [Proposition 6.5](#). They are identities of finite word evaluations before passage to cohomology; no formal Fox calculus over an infinite completed group is invoked. For each finite coefficient calculation one may replace ω_2 by any ordinary integer representative modulo the exponent of the finite semidirect, Heisenberg, or extraspecial group being evaluated; [Lemma 5.1](#) then shows that the resulting first and mixed second derivatives are independent of the representative.

Let G act on A and $A^\vee$, and evaluate a word in

$$G \ltimes (A \times A^\vee \times \mathbb{F}_2).$$

If $\bar{u}$ is its base value, $D_u, D_u^\vee$ are its two first derivatives, and β_u is its mixed central coordinate, then direct multiplication in the finite Heisenberg group gives

$$D_{uv} = D_u + \bar{u}D_v, \quad D_{u^{-1}} = -\bar{u}^{-1}D_u, \quad (41)$$

$$D_{u^g} = \bar{g}^{-1}D_u + \bar{g}^{-1}(\bar{u} - 1)D_g, \quad D_{u^n} = \sum_{j=0}^{n-1} \bar{u}^j D_u, \\ \beta_{uv} = \beta_u + \beta_v + (D_u^\vee)(\bar{u}D_v), \quad \beta_{u^{-1}} = \beta_u + (\bar{u}^{-1}D_u^\vee)(\bar{u}^{-1}D_u). \quad (42)$$

The formulas for conjugates, commutators, and powers follow recursively. When $\bar{u}$ has odd order e and the exponent is the 2-primary idempotent, the power sum is the norm projector

$$P = 1 + \bar{u} + \cdots + \bar{u}^{e-1}.$$

We now spell out the first-order ledger at a tame lower map. Write S, T for the lower actions of σ, τ , put $U = S^{\omega_2}$, and let (a, b, c, d) be the lift variables on (σ, τ, x_0, x_1) . The ω_2 -power rule gives

$$D(u_i) = P(Dx_i + D\tau), \quad (43)$$

and hence

$$D(d_0) = Pb + (P + 1)c, \quad D(z_0) = U^{-1}c. \quad (44)$$

The tame relator gives the unnormalized row

$$D(\tau^\sigma \tau^{-2}) = S^{-1}(1 + T)a + (S^{-1} + 1 + T)b. \quad (45)$$

The wild relator has the following first-order table:

factor	a	b	c	d
h_0	0	0	0	0
u_1^{-1}	0	P	0	P
x_1^σ	0	0	0	S^{-1}
$[d_0, z_0]$	0	0	0	0

Thus

$$L_w = Pb + (P + S^{-1})d. \quad (46)$$

Together, (45) and (46) are the Jacobian displayed in [Lemma 5.5](#).

The only non-obvious entry in this ledger is the first line. The variation of $g_0 = \sigma_2^2$ contributes only in order at least two, because the lower values of x_0 and d_0 at the tame map are both 1. If $P = 0$, then $Dd_0 = Dx_0 = c$, and the exact class-two identity of [Lemma 5.2](#) shows that h_0 has the same elementary and central shadow as x_0^2 . If $P = 1$, then $T = 1$ and the 2-primary operator U acts trivially on the simple characteristic-2 quotient; its lift to the Heisenberg or extraspecial coefficient group is a central automorphism of order at most 2, so $g_0 = \sigma_2^2$ acts trivially on the full coefficient group. Again [Lemma 5.2](#) applies. This is the precise reason that h_0 may be replaced by x_0^2 in the linear, mixed cup, and determinant calculations.

Write b_{cl} for the bilinear commutator pairing of the chosen class-two coefficient group. For the mixed Heisenberg and extraspecial calculations on normalized representatives supported on x_0 , the ledger is

factor	V-coordinate	central coordinate
h_0	0	$\lambda(c)$ in $H(A)$, or $q(c)$ for κ_q^0
u_1^{-1}	0	0
x_1^σ	0	0
$d_0 = (x_0\tau)^{\omega_2}x_0^{-1}$	$(P+1)c$	*
$z_0 = x_0^{\sigma^2}$	$U^{-1}c$	*
$[d_0, z_0]$	0	$b_{\text{cl}}((P+1)c, U^{-1}c)$

The starred central coordinates of d_0 and z_0 are immaterial because these words occur only inside the commutator. The table gives both the mixed Hessian in [Lemma 5.14](#) and the exact base determinant word expansion

$$Q_A^0(c) = q(c) + b_q((P+1)c, U^{-1}c)$$

of [Proposition 6.5](#). The starred entries may contain the non-diagonal action corrections m_c from [Lemma 6.1](#); they are central coordinates of d_0 and z_0 and hence do not affect the commutator. No relator-by-relator Stokes identity is used: the correction terms in the finite-word Stokes formula cancel only after summing the tame and wild relators, exactly as in [Proposition 5.8](#).

Machine-readable serialization We serialize the profinite exponent and the final relator as the finite integer-and-word block used by the independent computation. This record is provided for reproducibility and is not an input to the mathematical proof. The profinite exponent ω_2 is used throughout the proof itself. This appendix contains only the finite integer representative and exact word block used by the independent computation mentioned in [Remark 1.3](#).

For the finite verification modulus

$$M_{\text{ver}} = 85667662080 = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23,$$

the profinite idempotent $\omega_2 \in \widehat{\mathbb{Z}}$ has residue

$$\omega_2 \equiv 40491355905 \pmod{M}_{\text{ver}}.$$

Thus, in finite groups whose exponents divide M_{ver} , the exponent ω_2 may be serialized by the integer 40491355905. The proof itself always uses the genuine profinite exponent. The corresponding machine-readable block is

sigma,tau,x0,x1

```

sigma2 = sigma^40491355905
g0 = sigma2^2
u0 = (x0*tau)^40491355905
u1 = (x1*tau)^40491355905
d0 = u0*x0^-1
z0 = x0^sigma2
c0 = d0^-1*z0^-1*d0*z0
dg = d0^g0
hcomm = dg^-1*d0^-1*dg*d0
h0 = (x0^g0)*x0*dg*d0*d0^2*hcomm

```

```

h0*u1^-1*(x1^sigma)*c0

```

The tame relation $\tau^\sigma = \tau^2$ is implicit and is not repeated in the relations section.

The preceding block records the exact word used in the computational verification. The computation is not used in the proof of the marked normalization or any other mathematical step.

Proof dependency certificate This appendix gives a compact dependency certificate for the proof. The table pairs each load-bearing input with its first downstream use, making the route from the finite-word and cohomological calculations to the boundary-framed theorem explicit. The preceding sections prove the inputs needed by the induction and reconstruction step. The table records each load-bearing result and its first downstream use; no omitted calculation or computational verification is hidden in the implication chain.

Proved input	First downstream use
Proved input	First downstream use
Finite-word Stokes identities, Lemma 5.7 , Proposition 5.8	Fox–Heisenberg chain map
Exact cone dévissage and simple normal forms, Lemma 5.11 , Lemma 5.13	duality for arbitrary elements
Normalized cochain conventions and explicit section/transversal homotopies, Appendix 12	literal normalization of (0)
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